# Place Value

Each Place in a number has a unique Value
Value of a digit is determined by it's place.

our base ten system uses TEN Digits

0 1 2 3 4 5 6 7 8 9

Each Place is a POWER of ten.

Whole Numbers

Parts of Wholes or Fractions

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1000's | 100's | 10's | 1's | • | $\frac{1}{10}$ | $\frac{1}{100}$ | $\frac{1}{1000}$ | ... |
| $10^3$ | $10^2$ | $10^1$ | $10^0$ | | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | |
| thousands | Hundreds | Tens | ones | | tenths | hundreths | thousanths | |

## Types of Numbers

Whole numbers are the natural Numbers including 0.
Natural Numbers are the counting numbers and don't include 0.

Integers are positive and Negative whole Numbers.

Consecutive Numbers follow in sequence
$n, n+1, n+2, n+3 \ldots$

EVEN Number CAN be broken in two equal parts with No remainder.
ODD Numbers CAN'T be broken as such.

Rational numbers are numbers that CAN be expressed as a ratio of two integers.

Examples of rational numbers $\frac{2}{3}$, 1.07, $-3$

Irrational numbers CAN'T be expressed as a ratio of two integers.

Examples of irrational Numbers $\pi$, e, $\sqrt{2}$

THERE is an infinite SET of irrational Numbers.

REAL numbers are all the numbers that fit on a number line.

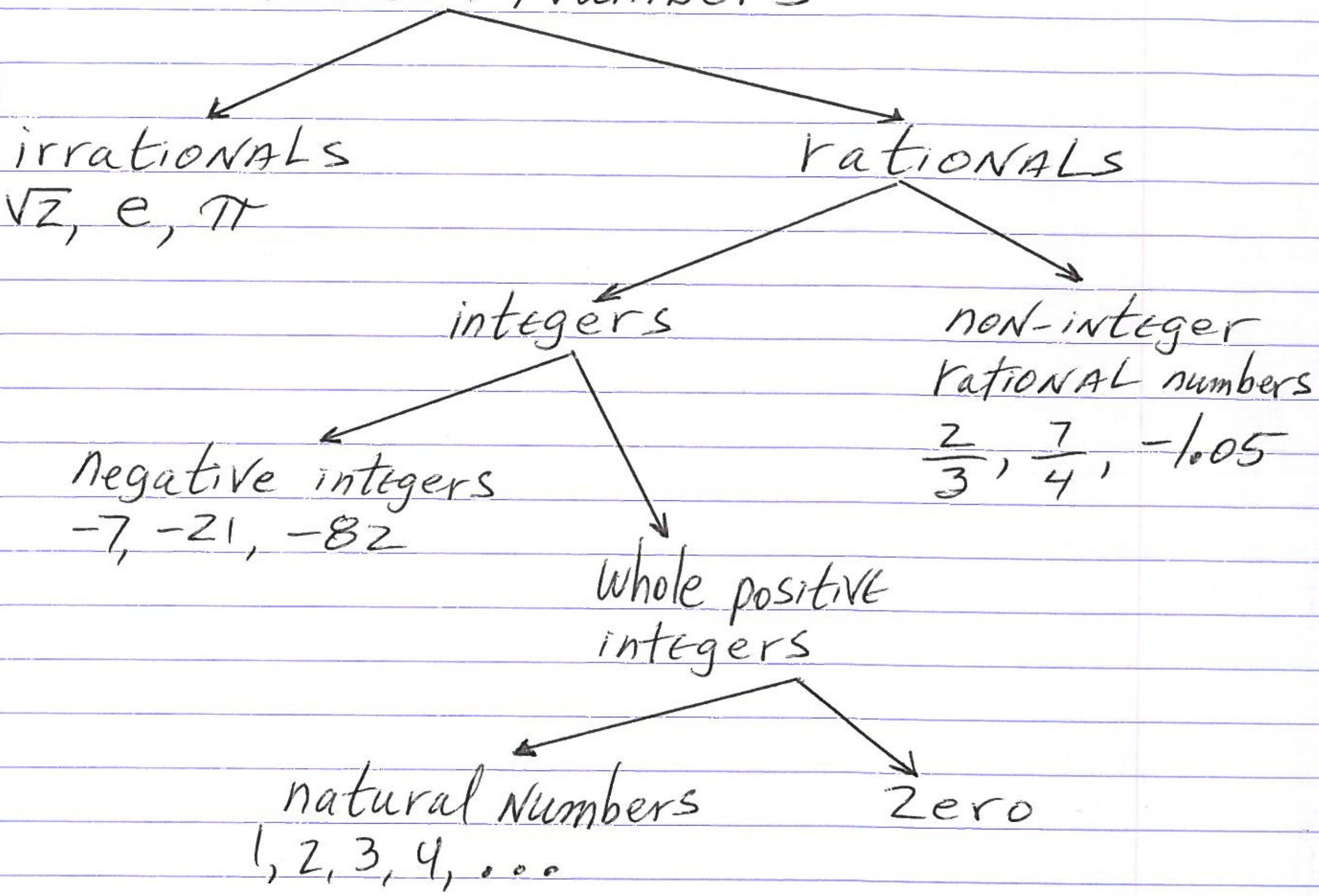
- Real numbers include both rational and irrational numbers.
  - $\sqrt{-1}$ and 3/0 are not real
- irrational numbers can't be pinpointed on a number line, but they are there.

Imaginary Numbers are square roots of negative numbers $i = \sqrt{-1}$

$i^2 = -1$

Complex numbers are combinations of real and imaginary numbers.

# Real Numbers



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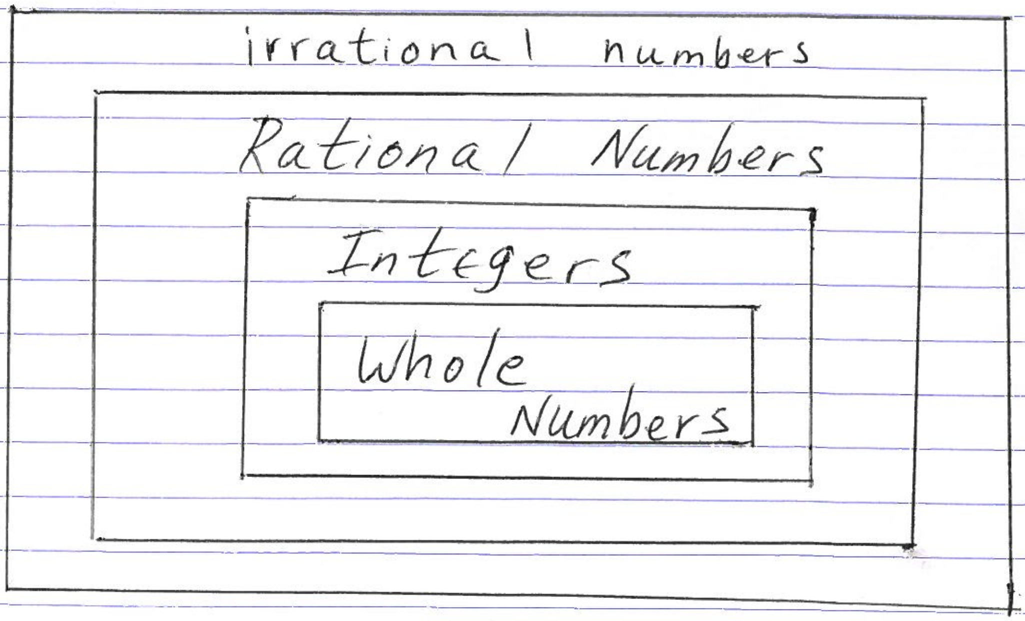
# Set of real numbers



# Basic OperatioNS

You can do four basic operations w/ number.

| | | |
|---|---|---|
| + | Addition | addend + addend = Sum |
| − | Subtraction | Minuend − Subtrahend = Difference |
| X | Multiplication | FACtor × FACTOR = Product |
| ÷ | Division | Dividend ÷ Divisor = Quotient |

Even numbers can be broken in two parts with nothing leftover ...−4, −2, 0, 2, 4...

Odd numbers have an "odd" remainder leftover when broken in Two ...−3, −1, 1, 3...

| ± | odd | EVEN |
|---|---|---|
| odd | EVEN | odd |
| EVEN | odd | EVEN |

| X | odd | EVEN |
|---|---|---|
| odd | odd | EVEN |
| EVEN | EVEN | EVEN |

## ORDER OF OPERATIONS

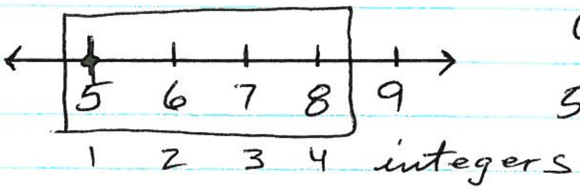
Parenthesis - Clean up operation in here 1ST
Exponents and roots from left to right
MultiplicATION
DiVISION
Addition
SubsTraction

Work from left to right doing × AND ÷ 1ST, then + AND − Last.

To Add a positive Number and a Negative Number — ignore the signs, take the difference of the Two Numbers and add the sign of the larger Number.

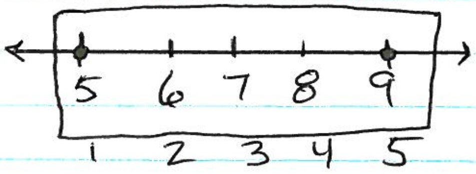
# Counting the number of integers between integers

- If exactly one endpoint is included, simply subtract. The difference is the answer



9 − 5 = 4 if 5 is included.

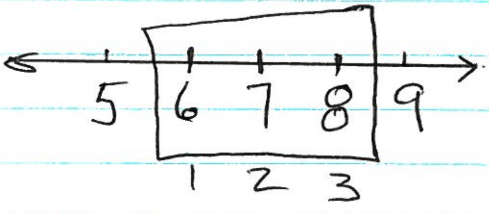
- If both endpoints are included, find difference and add one.



9 − 5 = 4 if both 5 and 9 included

4 + 1 = 5

- If neither endpoints are included, find difference then subtract one



9 − 5 = 4 if neither 5 nor 9 included

4 − 1 = 3

Consecutive integers are in continuous sequence. If m is 1st number, then m + 1 is second, m + 2 is 3rd, . . .

For even or odd consecutive sequences starting with m, use m + 2, m + 4, m + 6 . . .

# Number Theory

## Fundamental Theorem of ARITHMETIC

- Every Composite Number can be expressed as a product of prime Numbers in exactly one way.

Positive integers are either Prime or Composite

Prime Numbers are positive integers that have exactly Two factors — one and itself. The Set of prime integers is infinite.

Composite numbers have more than one and itself as factors.

Factors — a factor of a number is an integer that divides evenly into another with No remainders
- Also called divisors
- Composite Numbers have a finite set of integers factors

Multiples — are the products of a given Number created when multiplied by another integer.
- Every integer has an infinite set of multiples.

Prime factorization
- All Composite Numbers are "Composed" of products of prime Numbers.

# Divisibility Rules

When one number is divided into another with no remainders, we say that number is divisible by the other number.

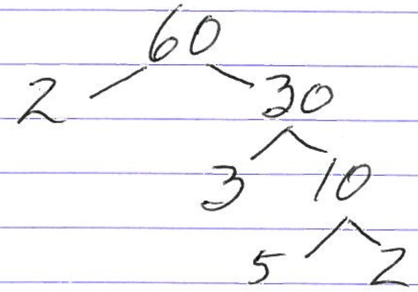
If x is Not divisible by y, then there will be a remainder

## Divisibility Rules for numbers 0-10

0 → No Number is divisible by 0
1 → Every Number is divisibly 1
2 → All even Numbers are divisibly by 2
Even numbers end in 0, 2, 4, 6 or 8.
3 → If the sum of the digits is divisible by 3, then the whole Number is divisible by 3
4 → If the last two digits are divisible by 4
5 → If the last digit is 0 or 5.
6 → Must be even and divisible by 3
7 → No simple rule. Here it is anyhow
Is 2,513 divisible by 7? First, multiply ones digit by 2, subtract this product from the non-ones digit numbers. If this difference is divisible by 7, then the whole number is
EG. 3×2=6  251-6=245÷7=35
8 → If the last 3 digits are divisible by 8
9 → Sum of digits is divisible by 9
10 → Number ends in 0

To find Prime factorization, keep dividing the number until every integer divisor is prime.

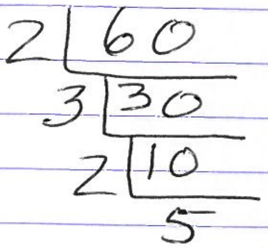
Examples of Prime factorization



$60 =$

$2 \cdot 3 \cdot 2 \cdot 5 =$

$2^2 \cdot 3 \cdot 5$



## Greatest Common FACTOR or GCF

- the largest factor that two or more numbers share

Prime factorization Method to find GCF

① do prime factorization of All Numbers
② determine which primes are common
③ Select Common primes to **LOWEST** degree.
④ multiply all the lowest degreed Common prime factors. Product is GCF.

Example: Find GCF of 60, 45 and 90

60 → $2^2 \cdot 3 \cdot 5$

45 → $5 \; 3^2$

90 → $2 \cdot 3^2 \cdot 5$

2 is Not common to All three, but 3 and 5 are

Take Common factors to lowest degree and Multiply

$3 \cdot 5 = 15 = GCF$

You can also list all the factors of Numbers and find GCF from list

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 30 | 1, | 2, | 3, | 5, | 6 | | 10, | 15, | | 30 | | |
| 45 | 1, | | 3 | 5 | | 9 | | 15 | | | 45 | |
| 90 | 1, | 2, | 3, | 5, | 6, | 9, | 10 | 15, | 18 | , 30, | 45, | 90 |

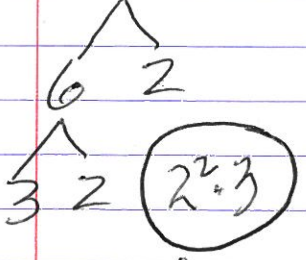
15 is the greatest factor common to All 3 #'s

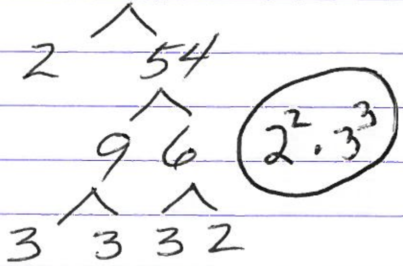
Finding LEAST Common Multiple (LCM) using Prime factorization.

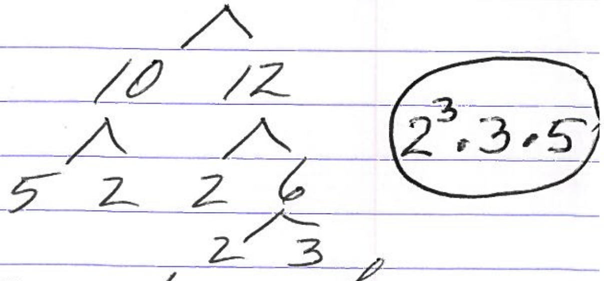
-USE SAME method as GCF, but this Time CHOOSE every unique prime factor to the LARGEST degree and multiply

Find LCM of 12, 108 and 120

$12 = 2^2 \cdot 3$ $\quad 108 = 2 \cdot 3^3$ $\quad 120 = 2^3 \cdot 3 \cdot 5$







Each unique prime is 2, 3 and 5. The highest degree of each is $2^3$, $3^3$ and $5^1$
Therefore the LCM is

$$2^3 \cdot 3^3 \cdot 5 = 8 \cdot 27 \cdot 5 = 1{,}080$$

You can also make a list of all the multiples and find the least, but that can be time consumming

ⓐ

# Division by Primes method of finding LCM.

Find LCM of 12, 108 and 120

| | | | | |
|---|---|---|---|---|
| 2 | 12 | 108 | 120 | Find lowest common prime and divide out |
| 2 | 6 | 54 | 60 | Keep going |
| 3 | 3 | 27 | 30 | If there's any common take out factors & bring down the others |
| 2 | 1 | 9 | 10 | |
| 3 | 1 | 9 | 5 | ⇄ Last line should be All primes |
| | 1 | 3 | 5 | |

Multiply All the outside Numbers

$2 \cdot 2 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 5 = 1080 = LCM$

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ONE MORE METhod of finding LCM.

- Find the Multiples of the largest Number, then select the smallest that is also a multiple of the smaller numbers

FIND LCM of 6, 15, 18

Largest # Multiples 18, 36, 54, 72, 90

You KNOW 15's Multiples end with 0 or 5

Is 90 a multiple of 15? Yes, ∴ LCM

$$\begin{array}{r|lll} 2 & 10 & 12 & 21 \\ \hline 3 & 5 & 6 & 21 \\ \hline & 5 & 2 & 7 \end{array}$$

$2 \times 3 \times 5 \times 2 \times 7 = 420$

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$$\begin{array}{ccc} 10 & 12 & 27 \\ \wedge & \wedge & \wedge \\ 5 \quad 2 & 2^2 \times 3 & 7 \quad 3 \end{array}$$

Unique prime to highest degree

$2^2 \times 3 \times 5 \times 7 = \underline{420}$

Product of LCM and GCF of the Numbers is equal to the product of the numbers themselves.

Example: 18 24

$$\begin{array}{cc} 18 & 24 \\ \wedge & \wedge \\ 2 \cdot 3^2 & 2^3 \cdot 3 \end{array}$$

GCF = Common factors to <u>lowest</u> degree

$= 2 \cdot 3 = 6$

LCM = unique prime factors To <u>highest</u> degree

$= 2^3 \cdot 3^2 = 72$

$18 \cdot 24 = 432 = 6 \cdot 72$

You can find LCM of Two Numbers by taking the product and dividing out the GCF

Since $GCF \cdot LCM = A \cdot B$

A = 1st #
B = 2nd #

then $$\frac{GCF \cdot LCM}{GCF} = \frac{A \cdot B}{GCF}$$

$$LCM = \frac{A \cdot B}{GCF}$$

Some numbers are relatively prime → they have No common prime Number

15 and 14 are relatively prime

# Number Theory (cont.)

Every number is divisible by itself
No number has a divisor greater than itself

Remainder is always smaller than divisor

Number thoory is used extensively with manipulating fractions

Fractions are simply parts of whole numbers and are represented as the spaces between integers on the number line.

$$\frac{A}{B} = A \div B \qquad \frac{A \longrightarrow \text{numerator}}{B \longrightarrow \text{denominator}}$$

When adding fractions, denominators must be equal. You find common denominators by finding the LCM or just multiplying the denominators.

$$\frac{A}{B} \pm \frac{C}{D} = \frac{AD \pm BC}{BD}$$

When you multiply a fraction, just multiply across numerators and denominators

$$\frac{A}{B} * \frac{C}{D} = \frac{AC}{BD}$$

When dividing fractions multiply by reciprocal of divisor.

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} * \frac{D}{C} = \frac{AD}{BC}$$

# Solving Complex Fractions

$$\frac{\frac{A}{B}}{\frac{C}{D}} = \frac{A}{B} \div \frac{C}{D} = \frac{A}{B} * \frac{D}{C} = \frac{AD}{BC}$$

Short cut → go directly to → $\frac{AD}{BC}$

or multiply by LCM

$$\frac{\frac{3}{4}}{\frac{5}{6}} = \frac{\frac{3}{4} \cdot 12}{\frac{5}{6} \cdot 12} = \frac{9}{10}$$

other method =

$$\frac{3 \cdot 6}{4 \cdot 5} = \frac{18}{20} = \frac{9}{10}$$

## Converting improper fractions to mixed numbers

$$x\,\frac{y}{z} = \frac{(z \cdot x) + y}{z}$$

Example $3\frac{3}{5} = \frac{15+3}{5} = \frac{18}{5}$

## Reducing fractions to lowest terms

→ divide out GCF from numerator and denominator

→ Example: $\frac{26}{91}$ GCF of 26 ~~&~~ 91 = 13

$$\frac{26 \div 13}{91 \div 13} = \frac{2}{7}$$

## Comparing fractions

- quickest way is to find cross products and compare the products

Example:

$$\frac{3}{4} \bigcirc \frac{5}{8}$$

$24 > 20 \therefore \frac{3}{4} > \frac{5}{8}$

If $\frac{a}{b} < 1$, then $a < b$

If $\frac{a}{b} = 1$, then $a = b$

If $\frac{a}{b} > 1$, then $a > b$

When $a > b$ in fraction $a/b$, its called an improper fraction. To convert improper fraction to mixed number, divide the numerator and denominator with the remainder becoming the numerator of the fraction part of a mixed Number.

example. $$\frac{23}{4} = 23 \div 4 = \begin{array}{r} 5 \\ 4\overline{)23} \\ \underline{20} \\ 3 \end{array} = 5\frac{3}{4}$$

Think: How many times does 4 go into 23? 5 times with 3 remaining.

also remember

$$\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a} = \frac{b+c}{a}$$

$$\frac{3+5}{4} = \frac{3}{4} + \frac{5}{4} = \frac{8}{4} = 2$$

# Basic Rules of Numbers

## Inverse Property

-For every real number $a$, $-a$ is its additive inverse such that

$$a+(-a)=0 \qquad \text{or}$$
$$a-a=0 \qquad a*-1=-a$$

To find INVERSE, multiply by $-1$

-For every real number $a$, $\frac{1}{a}$ is its multiplicative inverse such that,

$$a\cdot\frac{1}{a}=\frac{a}{a}=1$$

## Denseness Property

-For every real numbers $a$ and $b$, there exists a real number $c$ such that $a<c<b$

## Identity Principles

$$A+0=A \qquad A-0=A \qquad A\cdot 1=A$$

$$A\div 1=\frac{A}{1}=A \qquad (-A)=-1(A)=-1A$$

## Commutative Axiom of Addition & Multiplication

-The order of adding or multiplying doesn't matter if it's all addition or multiplication

$\longrightarrow$ $a+b+c=c+b+c$

$\longrightarrow$ $a\cdot b\cdot c=c\cdot b\cdot a$

## Zero Properties

$n - n = n + (-n) = 0$

$n * 0 = 0$ $\quad n \div 0 \Rightarrow$ undefined, but $0 \div n = 0$

If $a * b = 0$, then either $a = 0$ or $b = 0$.

## Distributive Property

$$a(b+c) = ab + ac = a(b+c)$$

## Fraction Rules

- a fraction is really a division problem.

$$\frac{a}{b} = a \div b = a \div \frac{b}{1} = a * \frac{1}{b} = \frac{a}{b}$$

$\frac{a}{a} = a \div a = \underline{1}$ Any number divided by itself equals one.

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{a \cdot b}{b \cdot a} = \frac{ab}{ab} = \frac{a}{a} \cdot \frac{b}{b} = 1 \cdot 1 = \underline{1}$$

The set of numbers above are called Reciprocals
Reciprocals are two numbers whose product is one.

If $a = a$, then $ab = ab$ and $\frac{a}{b} = \frac{a}{b}$

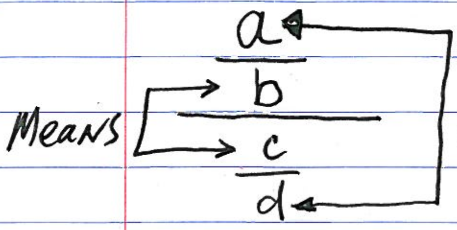
If $a = a$, then $a \pm b = a \pm b$

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b} = -1\left(\frac{a}{b}\right)$$

$a(bc) = abc \qquad a\left(\frac{b}{c}\right) = \frac{ab}{c}$

$$\frac{\frac{a}{b}}{c} = \frac{a}{bc} \qquad \frac{a}{\frac{b}{c}} = \frac{ac}{b} \qquad \frac{1}{\frac{b}{c}} = \frac{c}{b} \qquad \frac{\frac{1}{a}}{b} = \frac{1}{ab}$$

Complex fractions are numbers with a fraction as a numerator and a fraction as a denominator.



$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\overset{\text{Product of Extremes}}{ad}}{\underset{\text{product of means}}{bc}}$$

Example: $\frac{2}{3} \div \frac{1}{6} = \frac{\frac{2}{3}}{\frac{1}{6}} = \frac{2 \times 6 = 12}{3 \times 1 = 3} = 4$

old fashioned way $\frac{2}{3} \div \frac{1}{6} = \frac{2}{3} \times \frac{6}{1} = \frac{12}{3} = 4$

Rule for simplifying complex fractions

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc} \quad \text{and} \quad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

## Adding and subtracting fractions

$\frac{a}{b} \pm \frac{c}{d} = \frac{ad + bc}{bd}$ Eq. $\frac{3}{4} + \frac{2}{5} = \frac{(3)(5) + (4)(2)}{(4)(5)}$

$= \frac{15 + 8}{20} = \frac{23}{20}$

$$\frac{a + b + c}{d} = \frac{a}{d} + \frac{b}{d} + \frac{c}{d}$$

$$\frac{a - b}{c - d} = \frac{b - a}{d - c}$$

# More properties of fractions

Dividing by fractions will product larger numbers

Examples: $5 \div \frac{1}{2} = 5 * 2 = 10$ $\quad 10 > 5$

$\frac{1}{4} \div \frac{1}{3} = \frac{1}{4} * \frac{3}{1} = \frac{3}{4}$ $\quad \frac{3}{4} > \frac{1}{4}$

Multiplying by fractions will product smaller numbers

Examples: $10 * \frac{1}{2} = \frac{10}{2} = 5$ $\quad 5 < 10$

$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ $\quad \frac{1}{8} < \frac{1}{2}$

## Additional Properties of numbers

$a + b = b + a$ and $a \cdot b = b \cdot a$ but

$a - b \neq b - a$ and $a \div b \neq b \div a$

$$a - b = -1(b - a) = -b + a$$

$$-(a - b) = b - a$$

$$-(a + b) = -a - b = -b - a$$

$$a - (b + c) = a - b - c$$

$$a - (b - c) = a - b + c$$

$$1 - x = -x + 1$$

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# Inequalities

The same rules for equalities apply to inequalities except for applications involving multiplying or dividing negative numbers.

If $A < B$, then $AC < BC$ if $C > 0$

If $A < B$, then $\frac{A}{C} < \frac{B}{C}$ if $C > 0$

If $A < B$, then $AC > BC$ if $C < 0$

If $A < B$, then $\frac{A}{C} > \frac{B}{C}$ if $C < 0$

The inequality sign is reversed if both sides of equation are multiplied or divided by a negative number

If $a > b$, then $a - b$ is positive

If $a < b$, then $a - b$ is negative

If $a$ and $b$ are both positive or both negative and $a < b$, then $\frac{1}{a} > \frac{1}{b}$ $\frac{1}{3} < \frac{1}{4}$

If $ab > 0$ and $a < b$, then $\frac{1}{a} > \frac{1}{b}$

example: $4 \cdot 8 = 32$

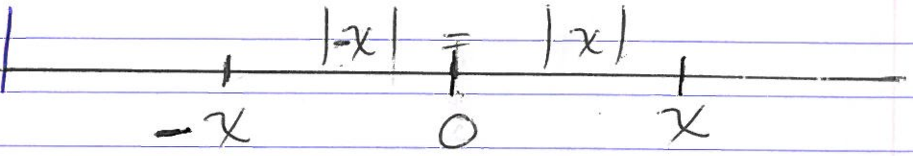
$4 < 8$ $\frac{1}{4} > \frac{1}{8}$

# Absolute Value

Absolute Value means the actual distance between numbers No matter if they are positive or Negative numbers

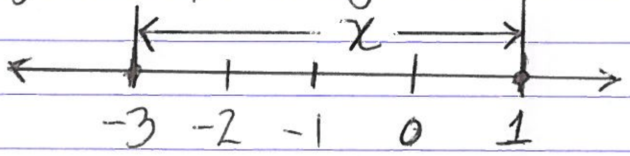
the Symbol $|x|$ means the distance between the inverses of $x$

$|-x| = |x|$



the distance between. 0 and $-x$ is the same. as the distance between 0 and $x$.

It doesn't matter if distance is positive or Negative, its just distance.



$x = |-3-1| = |-4| = 4$

Absolute Value can be used when adding or Subtracting Positive and Negative Numbers

$+A + (-B) = |A| - |B|$ then take on sign of larger Absolute Value.

Rules $|-x| = |x|$ $\quad |xy| = |x| \cdot |y|$ $\quad |a|^2 = a^2$

$|x \pm y| = |x| \pm |y|$

$$|x| = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -x \text{ if } x < 0 \end{cases}$$

$$|x| = \sqrt{x^2}$$

$$|x-y| = |y-x|$$

# Solving Inequalities with Absolute Value

1) Solutions of $|a|<b$ satisfy $-b<a<b$

- The conjunction is read "a is both greater than $-b$ and less than $b$."

Solving for $x$ in problems of form

$|ax+b|<c$ By definition, this equation means:

$$-c<ax+b<c$$

$-c<ax+b<c$ Solve for $x$

$-b \qquad -b \quad -b$ subtract $b$ from both sides

$$-c-b<ax<c-b$$

$$\frac{-c-b}{a}<\frac{ax}{a}<\frac{c-b}{a}$$ Divide all sides by a

$$\underline{-c-b}<x<\frac{c-b}{a}$$ Final Solution

Example: $|3x+4|<5$ Set up conjunction

$-5<3x+4<5$

$-4 \qquad -4 \quad -4$ Subtract 4 from All sides

$-9<3x<1$

$$\frac{-9}{3}<\frac{3x}{3}<\frac{1}{3}$$ Divide 3 on All sides

$-3<x<\frac{1}{3}$ So $x$ is both greater than $-3$ and less than $1/3$

$x>-3$ AND

$x<\frac{1}{3}$

# Solving Inequalities with Absolute Value (cont.)

2) Solutions of $|a| > b$ satisfy $a < -b$ or $a > b$

Solving for $x$ in problems of form

$|ax+b| > c$ By definition, this equation means:

$$ax+b < -c \quad \text{OR} \quad ax+b > c$$

Solve for x
Subtract b

$$\begin{array}{rcl} ax+b & < & -c \\ -b & & -b \\ \hline ax & < & -c-b \end{array} \qquad \begin{array}{rcl} ax+b & > & c \\ -b & & -b \\ \hline ax & > & c-b \end{array}$$

Divide a

$$\frac{ax}{a} < \frac{-c-b}{a} \qquad \frac{ax}{a} > \frac{c-b}{a}$$

Solution

$$x < \frac{-c-b}{a} \quad \text{OR} \quad x > \frac{c-b}{a}$$

Example: $|3x+4| > 5$

$$3x+4 < -5 \quad \text{or} \quad 3x+4 > 5$$

$$\begin{array}{rcl} 3x+4 & < & -5 \\ -4 & & -4 \\ \hline 3x & < & -9 \end{array} \qquad \begin{array}{rcl} 3x+4 & > & 5 \\ -4 & & -4 \\ \hline 3x & > & 1 \end{array}$$

$$\frac{3x}{3} < \frac{-9}{3} \qquad \frac{3x}{3} > \frac{1}{3}$$

$$x < -3 \quad \text{or} \quad x > \frac{1}{3}$$

# Another Way to Solve inequalities of form $|ax+b| < c$ and $|ax+b| > c$

the big difference is to think "Both... And" when the absolute value is less than the constant and to think "Either... or" when the absolute value is greater than constant.

Simply split the Absolute Value using the definition of A.V. $|x| = x$ when $x > 0$

$-x$ when $x < 0$

| $\lvert ax+b\rvert < c$ | $\lvert ax+b\rvert > c$ |
|---|---|
| Example: $\lvert 3x+4\rvert < 5$ | $\lvert 3x+4\rvert > 5$ |
| Split into 2 problems | Split into 2 problems |
| $3x+4<5$ and $-(3x+4)<5$ | $3x+4>5$ or $-(3x+4)>5$ |
| $3x+4<5$    $-3x-4<5$ | $3x+4>5$    $-3x-4>5$ |
| $-4\quad -4$    $+4\quad +4$ | $-4\quad -4$    $+4\quad +4$ |
| $3x<1$    $-3x<9$ | $3x>1$    $-3x>9$ |
| $\frac{3x}{3} < \frac{1}{3}$    $\frac{-3x}{-3} > \frac{9}{-3}$ | $\frac{3x}{3} > \frac{1}{3}$    $\frac{-3x}{-3} > \frac{9}{-3}$ |
| $x < \frac{1}{3}$ and $x > -3$ | $x > \frac{1}{3}$ OR $x < -3$ |

# Decimals

Decimals are fractions expressed exclusively in the base ten number system. The denominator in a decimal is always a power of 10.

To convert any fraction to decimal divide denominator into numberator.

Since $\frac{A}{B} = A \div B = B\overline{)A}$ then

$$\frac{3}{4} = 3 \div 4 = 4\,\overset{0.75}{\overline{)3.00}} = 0.75$$

$$\begin{array}{r} 0 \\ \hline 30 \\ -28 \\ \hline 20 \\ 20 \\ \hline 0 \end{array}$$

Adding and subtracting decimals is just like adding or subtracting any base ten whole number. Just make sure you line up the decimal points.

Example $0.073 + 2.013 = \begin{array}{r} 2.013 \\ +\,0.073 \\ \hline 2.086 \end{array}$

When multiplying decimals, you don't need to line up decimal points, multiply the numbers then total up the decimal places in factors and the sum is the number of decimal places in the product.

$$0.0007 \times 0.003 = \begin{array}{rl} 0.0007 & \text{4 Places} \\ \times\ 0.003 & \text{3 Places} \\ \hline 0.0000021 & \text{7 Places} \end{array}$$

When dividing decimals, the divisor must be an integer when not using a calculator.

$25 \div 0.5 \Rightarrow 0.5\overline{)25}$

dividend ⟶ ↑ ← divisor

To change divisor to integer, shift the decimal point to right until its an integer shift decimal point in dividend the same number of decimal spaces. You're basicly dividing by the same multiple of 10.

$0.5\overline{)25.0} \Rightarrow 5\overline{)250}$

When you multiply a decimal by ten's, move decimal point to the right by the number of 0's in the power of ten.

Example $0.000607 \times 1000 = 0.607$

When divide a decimal with a divisor that is a power of ten, move the decimal point to the LEFT by the number of 0's in the power of 10.

example: $0.000607 \div 1000 = 0.000000607$

Calculators have made manipulating decimals much easier, but sometimes its faster and easier to use pencil and paper when dealing with operations of decimals.

# Converting Fractions to decimals and vice versa.

Remember a fraction is a division problem

$$\frac{3}{8} = 3 \div 8 = \begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ \underline{00} \\ 30 \\ \underline{-24} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

think: How many times does 8 go into 3?

To convert 0.375 back to fraction, remove decimal point and the power of 10 in denominator is the Number of decimal place

$$0.375 = \frac{375}{1000} \leftarrow \text{3 zeros 3 decimal places}$$

$$\frac{375 \div 125}{1000 \div 125} = \frac{3}{8}$$

If you can easily convert the denominator into a power of ten, then do it

example: $\frac{6}{25} = \frac{6 \times 4}{25 \times 4} = \frac{24}{100} = 0.24$

To find a particular digit in a repeating decimal, Note the number of digits in the cluster that repeats. If there are 3 repeating digits, then every 3rd digit repeats

Example $\frac{1}{27} = 0.037037\ldots$ To find 50th digit, divide 50 by 3 to get 48, the 48th digit is 7 the 49th is 0 and the 50th is 3.

# Percents

Percents are a specific type of fraction. Per cent literally means "for every one hundred" Percents are fractions that always have 100 in the denominator.

Questions may be framed as "4 is what percent of 20?

- 2 ways to solve
  - Direct translation

$4 = \frac{x}{100} * 20$ solving for $x$, you get 20.

  - Set up proportion

$$\frac{4}{20} = \frac{x}{100} \Rightarrow \begin{aligned} 100 \cdot 4 &= 20 \cdot x \\ 20 &= x \end{aligned}$$

Questions may also be framed as "Sales tax on a \$140.00 item is \$2.20. What is the sales tax rate?"

$$\$\,40.00 \cdot \frac{x}{100} = \$\,2.20$$

$$\frac{100}{40} \cdot \frac{40x}{100} = \frac{\overset{1.10}{\cancel{2.20}}}{1} \cdot \frac{\overset{5}{\cancel{100}}}{\underset{1\cancel{2}}{\cancel{40}}} = 5.5$$

$$x = 5.5 \Rightarrow 5.5\%$$

# Changing from fractions to decimals To Per cents and back again.

Since percents are decimals and decimals are fraction, you can interchange them all depending on the application you need.

$$\frac{3}{4} = 3 \div 4 = 4\overset{0.75}{\overline{)3.00}} = \frac{75}{100} = 75\%$$

or

0.75

All of the above expressions are the same.

To change ANY decimal to a percent, move the decimal point to the RIGHT two places

$0.0038 = 0.38\%$ $\quad$ $3.8 = 380\%$

To change from Per Cent To decimal fraction, move the decimal point To the LEFT Two places. and drop the per cent symbol

$3.6\% = 0.036 = 36/1000$

If you want to go direct To fraction from a per cent, drop the per cent sign and put whatever is left OVER 100 in the denominator

$$62\% = \frac{62}{100} \qquad 6.2\% = \frac{6.2}{100} \cdot \frac{10}{10} = \frac{62}{1000}$$

# Percent Applications

What percent of 60 is 12?

Translate $$\frac{x}{100} * 60 = 12$$

Solve $$\frac{5}{3} \cdot \frac{x}{\begin{smallmatrix}\cancel{100}\\ \cancel{10}\ 5\end{smallmatrix}} \cdot \begin{smallmatrix}\cancel{60}\\ \cancel{6}\\ 3\end{smallmatrix} = \cancel{12}^{4} \cdot \frac{5}{\cancel{3}_{1}}$$

or $60x = 12$
$x = \frac{1}{5} = 20\%$

$$x = 4 \cdot 5 = 20 \Rightarrow 20\%$$

Same Questions may be phrased

What Number is 20% of 60

Translate $$x = \frac{\cancel{20}^{4}}{\begin{smallmatrix}\cancel{100}\\ \cancel{10}\ \cancel{5}\ 1\end{smallmatrix}} * \cancel{60}\ \cancel{6}\ 3 = 12$$

$$A\% \text{ of } B = B\% \text{ of } A$$

$$30\% \text{ of } 100 = 100\% \text{ of } 30$$
$$30 = 30$$

$$20\% \text{ of } 10 = 10\% \text{ of } 20$$
$$2 = 2$$

# Percent Increase and Percent Decrease

When quantities change, we can find its percent change by asking:

The change is what % of original amount!

## Finding percent increase

- Discover the difference between original amount and resulting amount
- Divide the difference by the original amount

$$\% \text{ Increase} = \frac{\text{actual increase}}{\text{original amount}}$$

$$= \left(\frac{A-B}{B} * 100\right)\%$$

A = Amt. After change

B = Amt Before change

## Example Test Question

An item priced at \$50 now will cost \$51.75 next year due to inflation. What is the current rate of inflation per year?

A = After change = \$51.75
B = Before change = \$50.00

$$\left(\frac{51.75 - 50}{50} * 100\right)\% = \left(\frac{1.75}{50} * 100\right)\% = 3.5\%$$

# PERCENT DECREASE

$$\frac{\text{Actual decrease}}{\text{Original Amount}}$$

ex. Decrease from 100 → 80

$$100 - 80 = \frac{20}{100} = 20\%$$

80 is a 20% decrease from 100.

Real formula for % decrease $\left(\frac{B-A}{B} * 100\right)\%$

A ⇒ After change

B → Before change

To increase a number by $x\%$, multiply it by $(1 + x\%)$. What is 50 increased by 25%. $50(1+25\%) = 50(1.25) = 62.5$

To decrease a number by $x\%$, multiply it by $(1 - x\%)$.

Example: What is 50 decreased by 25%

$$50(1-25\%) = 50(1-0.25) = 50(0.75) = 37.5$$

$$\frac{\text{Difference}}{\text{Starting Point}}$$

# Ratio

Ratios are simply mathematical comparisons. Comparing means making connections, discovering similarities/differences or finding some relationship between things.

Percents are ratios. Percents compare everything to 100.

The phrase "the ratio of $x$ to $y$" is translated to:

$x : y$ or $\frac{x}{y}$ Since ratios look and really are fracti all the rules of fractions Apply To ratios

Given ratio $\frac{A}{B}$, the ratio of A to the whole equals $\frac{A}{A+B}$

Common problem → distributing amounts given a fixed ratio.

$1,000 is To be distributed in the ratio of 3: 4: 5. What do each of the three recipients get?

$3+4+5 = 12$ ∴ There are 12 Parts. $\frac{1000}{12} = 83.33$

1st gets 3 parts $= 83\frac{1}{3} \times 3 = 250$ 2nd gets 4 parts or $333\frac{1}{3}$ and 4th gets $5 \times 83\frac{1}{3} = 416\frac{2}{3}$

# Combined or "Middle" Ratios

You may encounter problems with ratios in the form $a:b:c$ or if $\frac{a}{b}$ and $\frac{b}{c}$, what is $\frac{a}{c}$?
They are framed as, "if a is to b and b is to c, what is a to c?"

Example: $\frac{a}{b} = \frac{7}{3}$ and $\frac{b}{c} = \frac{2}{5}$, Find $\frac{a}{c}$

Solution: you need to make the "middle" term (b) the same in both ratios.

One way to do this is to multiply <u>each</u> by the <u>other.</u>

$b_1 = 3 \qquad b_2 = 2$

multiply 1st ratio $\frac{7}{3}$ by the other b term

$$\frac{a}{b} = \frac{7}{3} \times \frac{2}{2} = \frac{14}{6} = \frac{a}{b} \text{ when multiplied by } \frac{2}{2}$$

multiply the 2nd ratio by the 1st b term

$$\frac{b}{c} = \frac{2}{5} \times \frac{3}{3} = \frac{6}{15} = \frac{b}{c} \text{ when multiplied by } \frac{3}{3}$$

Notice how the two b terms are now the same

$\frac{a = 14}{b = 6}$ as $\frac{6 = b}{15 = c}$ You can substitute and say that $\frac{a}{c} = \frac{14}{15}$

# More on "Middle" Ratios

Think of Middle ratios as relative ratios and not constant, when compared to different things, the middle Ratio may change, but the middle "thing" being compared stays the same.

**Rule for Middle Ratios**

If $\frac{a}{b_1}$ is to $\frac{b_2}{c}$, then $\frac{a * b_2}{c * b_1} = \frac{a}{c}$

Example: $\frac{A}{b_1} = \frac{7}{3}$ and $\frac{b_2}{c} = \frac{2}{5}$ Find $\frac{a}{C}$

In terms of a, $b_1 = \frac{3}{7}a$ Solving for b using cross Products

In terms of C, $b_2 = \frac{2}{5}c$ " "

Set $b_1$ and $b_2$ equal and subsitute

$b_1 = \frac{3}{7}a$

$b_2 = \frac{2}{5}c$

substitute

$$\frac{3}{7}a = \frac{2}{5}c$$

$$\frac{7}{3} \cdot \frac{3}{7}a = \frac{2}{5}c \cdot \frac{7}{3}$$

$$\frac{1}{c} \cdot a = \frac{14}{15}c \cdot \frac{1}{c}$$

$$\frac{a}{c} = \frac{14}{15}$$ Check

$b_1 = 3$

$b_2 = 2$

$$\frac{a * b_2}{c * b_1} = \frac{a}{c}$$

$$\frac{7 * 2}{5 * 3} = \frac{14}{15}$$

# Proportions

**Proportions** are equal ratios. When ratios are equal, their cross products are equal.

Example: $\frac{3}{4} = \frac{6}{8}$ because $\frac{3}{4} \times \frac{2}{2} = \frac{6}{8}$

Notice how the cross products are equal.

$$3 \times 8 = 4 \times 6$$

$$\boxed{\text{Rule: If } \frac{A}{B} = \frac{C}{D}, \text{ then } AD = BC}$$

Example: The ratio of blue eyed people to brown eyed people is 3:7. In a group of 210 people, how many are blue-eyed.

Solution: You need to know the ratio of blue eyed people to the entire population. You find the group by adding the individual elements. If 3:7, then 3+7 = 10. So, ratio of blue eyed to group is 3:10. Use cross product Rule to find out how many in group of 210.

$$\frac{3}{10} = \frac{x}{210}$$

x = how many blue eyes in group of 210.

$$3 \times 210 = 10 \times x$$

$$90 = x$$

# Variation → Direct and Inverse

Variation mathematically shows how Two or more variables are related. Variation descibes Relationship.

## Direct Variation

- A change in one Variable produces an absolute or Constant change in the other.
- The equation for direct variation looks very much lide the Slope formula. You Divide the variables To produce the Constant

$$\boxed{\frac{Y}{X} = K \quad \text{or} \quad Y = Kx}$$

K= Constant of Variation

- One Variable is the constant multiple of the other
- As one Value increases, the other Value increases. Decrease in one Variable produces a decrease in the other.
  - the change in each Variable is dependent on K.
- K can also be called "the Constant of Proportionality."

# Direct Proportions

When two elements $x$ and $y$ vary directly with each other, we see a constant or absolute increase in one as well as the other.

If $x$ varies directly with $y$, then $\frac{x}{y} = K$

$K$ is called the constant of variation

Example: The greater amount you buy, the greater the total cost.

Ten doughnuts cost $6.00. How much will 15 doughnuts cost?

Set up proportion $\frac{6}{10} = \frac{x}{15}$  $x = \$9.00$

OR, use $\frac{6}{10}$ as the constant of variation

the constant stays the same no matter what amount is purchased.

If you want 15 then $0.6 × 15 = $9.00
If you want 45, then $0.6 × 45 = $27.00

Other Examples of Direct Variation
- The longer you work, the greater your pay
- The longer the time travelled, the greater the distance traversed.
- The longer time a machine or person works, the more gets produced

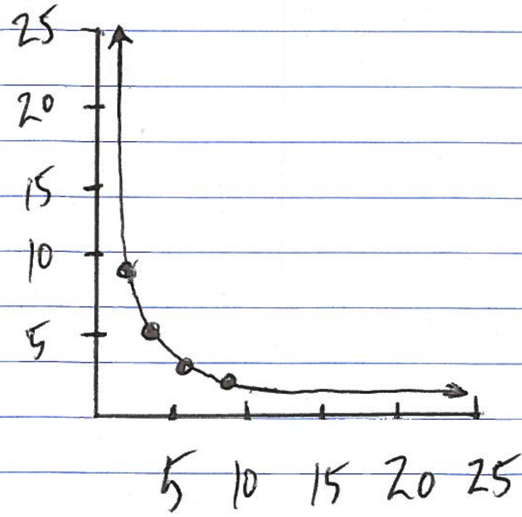
# Inverse Variation

- A change in one variable produces an inverse or opposite change in the other.
- The equation looks like a Multiplication problem.
- You Multiply the variables to produce the constant of variation.

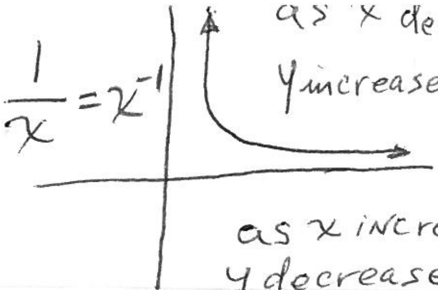
$$Y * X = K \quad \text{or} \quad Y = \frac{K}{X}$$

- Once again, K is the constant
- As one variable increases, the other variable has to decrease in order to maintain the constant value.
-

| Y | * | X | = | K or CONSTANT |
|---|---|---|---|---|
| 1 | * | 18 | = | 18 |
| 2 | * | 9 | = | 18 |
| 3 | * | 6 | = | 18 |
| 6 | * | 3 | = | 18 |
| 9 | * | 2 | = | 18 |
| 18 | * | 1 | = | 18 |



graph of $y = \frac{1}{x} = x^{-1}$



# Inverse Proportions

If x varies inversely with y, then $y = \frac{k}{x}$ or $K = xy$. If x increases, then y has to decrease in order for K to stay the same.

an example is when more machines or people do job, the less time it takes to do it. When speed increases, less time is needed. When volume increases, density decreases.

Example: Travelling at constant rate of 50 mph, a trip takes 2 hours. How long will the trip take if you travel at 60 mph?

You need to recognize that this is an inverse. As rate of travel increases, the time decreases. One way to solve is to find the constant of variation → the distance. Once you find distance, you can solve for time

$$D = 50 \text{ mph} \cdot 2 \text{ hours} = 100$$

$$100 = 60 \text{ mph} \cdot t$$

$$100/60 = t = 1\ ^2/_3 \text{ hours} = 1 \text{ hour } 40 \text{ min.}$$

another way is to "Invert" the proportion instead of straight $\frac{50}{60} = \frac{2}{x}$ invert $\frac{2}{x}$ to $\frac{x}{2}$

$$\frac{50}{60} = \frac{x}{2} \Rightarrow \frac{100}{60} = \frac{60x}{60} \Rightarrow x = \frac{5}{3} = 1\frac{2}{3}$$

# Rate Problems

Rates are really applications of ratio and proportion involving inputs and outputs. Common "inputs" include:

1) Time 2) money 3) work
Some investment is required

output is what you get after putting in the investment. outputs include:

1) Distance travelled 2) money spent
3) Work accomplished.

Most basic rate is the "UNIT" rate or how much, how many, how fast, .... for one. For unit rates we use the word "per" which means "for every"

If you can travel 120 miles in 4 hours, how fast are you going in one hour

$$\frac{120 \text{ miles}}{4 \text{ hours}} = \frac{x}{1 \text{ hour}} \qquad x = 30 \text{ miles } \underline{\text{per}} \text{ hour}$$

If x pencils costs y cents, how much does one pencil cost?

Solve for z

$$\frac{x}{y} = \frac{1}{z} \Rightarrow \frac{xz}{x} = \frac{1 \cdot y}{x} \Rightarrow z = \frac{y}{x}$$

# Example of Rate Problem

Andy drove the 200 miles to chicago at an average Speed of 10 mph faster than his regular average speed. If he completed the trip in 1 hour less than usual, what is his usual driving Speed in miles per hour?

Make a chart

| | DISTANCE | RATE | Time |
|---|---|---|---|
| Usual Trip | 200 miles | $r$ | $\frac{200}{r}$ |
| This trip | 200 miles | $r+10$ | $\frac{200}{r+10}$ |

Because the time for this trip is One hour less than the time for the usual trip, Solving for $\frac{200}{r+10} = \frac{200}{r} - 1$

Will product answer $\frac{200}{r+10} = \frac{200-r}{r}$

$$200r = (r+10)(200-r) = 200r - r^2 + 2000 - 10r$$

$$0 = -r^2 - 10r + 2000 = r^2 + 10r - 2000$$

$$0 = (r-40)(r+50)$$

Can't have Negative speed, So it's 40 MPH

# Work Problems

Work problems are rate problems.

R = Rate of work

$R = \frac{A}{T}$ A = Amount of work done
T = Time used To do work

$RT = 1$ ONE complete Job

When you do work problems, you are figuring the rate some person or some machine will work together or separately

Formula $\frac{1}{T_1} + \frac{1}{T_2} = \frac{1}{T_3}$

$T_1$ = hours worker/machine 1 takes To complete Job working alone

$T_2$ = hours worker/machine 2 takes to complete job separately

$T_3$ = hours workers or machines Needed when working Together.

The number ONE is used because problems refer to one whole or complete Job

Example: It would take Sam 3 hours to paint a room. It would take Joe twice as long to do it by himself. How long would it take both of them working together?

Solution: Let $T_1$ = Sam's Time to do work
$T_2$ = Joe's Time to do work
$T_3$ = Time together

If Sam can do a Job in 3 hours, then in one hour he can do $\frac{1}{3}$ of the Job. Joe can do $\frac{1}{6}$ of the job — he's taking Twice as Long

$$\frac{1}{3}+\frac{1}{6}=\frac{1}{T_3} \Rightarrow \frac{2}{6}+\frac{1}{6}=\frac{1}{T_3}$$

It Will take
Both of them 2 hours
Working together

$$\frac{3}{6}=\frac{1}{T_3}$$

$$3t_3 = 6$$

$$T_3 = 2$$

Example: One machine can do 5 loads in 60 minutes. Machine 2 can do 3 loads in 30 minutes. How many loads can be done if both machines are working for 45 minutes.

$$\frac{5 \text{ loads}}{60 \text{ minutes}} + \frac{3 \text{ loads}}{30 \text{ minutes}} = \frac{x}{45 \text{ minutes}}$$

$$\frac{5}{60}+\frac{6}{60}=\frac{x}{45}$$

$$\frac{11}{60}=\frac{x}{45}= \quad 45 \ast 11 = 60x$$

$$8.25 = x$$

# More on Work problems

Work problems deal with TWO or more workers doing the same job at different rates. The aim is to find out how long it will take workers to do a Job if the number of workers is increased or decreased or if they work at different rates. The same principles for work Problems also apply to other rate problems such as filling tanks, swimming pools, pipes, etc.

The ~~biggest~~ best Tactic is To express the <u>fraction</u> of the job that can be done in <u>ONE</u> unit of time

Example: If 2 people can do a job in 12 minutes and 1 person can do the Job in 30 min, how long does it take the other person To do the Job alone.?

Think → How much of the job can they do in <u>ONE MINUTE</u>. TWO can do 1/12 of the job in 1 minute

A person alone can do 1/30 of the Job in ONE minute. We don't know what the other person can do alone, So we express this a 1/x.

The problem said together. they can do the job at a rate of

1/12 of the Job per minute together.

Since together means add, we'll add their two rates individually and they will TOTAL $\frac{1}{12}$.

KNOWN

UNKNOWN

SOLVE for UNKNOWN

$$\frac{1}{30} + \frac{1}{x} = \frac{1}{12}$$

$$\frac{60x}{30} + \frac{60x}{x} = \frac{60x}{12}$$

multiply by 60x to get rid of x in denominator 60 because its a multiple of 12 and 30

$$2x + 60 = 5x$$
$$60 = 3x$$
$$20 = x$$

Since $x = 20$, then the other person can do $\frac{1}{20}$ of the job in one minute.

If he can do 1/20 of the Job in one minute, then he can do the whole Job in 20 minutes.

---

MORE Examples: A does a job in six days, B does the same job in 3 days How long will it take both of them together?

think → What can they do in ONE DAY

$$\underset{}{\overset{A}{\frac{1}{6}}} + \overset{B}{\frac{1}{3}} \Rightarrow \frac{1}{6} + \frac{2}{6} = \frac{3}{6} = \frac{1}{2}$$

they can do 1/2 of the job in one day So the job will take 2 DAYS

# Work Problems (Cont.)

If you're given a fraction of a job that can be done in a day, then the reciprocal will be the length of the entire Job

Note: UNIT RATE IS RECIPROCAL OF THE TOTAL AMOUNT OF TIME FOR THE ENTIRE JOB.

Example: A + B together can finish in 4½ days. B alone can do it in 10 days. How long will it take for A to do it alone?

Answer: It takes them 4½ days or 9/2 to finish so 2/9 can be done in one day. B can do 1/10 in one day.

If you take away B's rate from their combined rate, you will find A's rate → for one day.

$$\frac{2}{9}-\frac{1}{10}=\frac{20}{90}-\frac{9}{90}=\frac{11}{90} \quad \text{So } \frac{90}{11} \text{ or } 8\frac{2}{11} \text{ days}$$

---

Pump 1 fills Pool in 20 min. → $\frac{1}{20}$ of Pool in 1 minute
Pump 2 fills pool in 30 min. → 1/30 of Pool in 1 minute
Pump 3 fills pool in 10 min. → 1/10 of Pool in 1 minute

How long will it take all 3 to fill the pool

$$\frac{1}{20}+\frac{1}{30}+\frac{1}{10}=\frac{11}{60}$$

Reciprocal will tell how long to fill the pool $\frac{60}{11}$

Principle x Rate x Time = Interest

Distance = Rate x Time

# Mixture Problems

The ACT will sometimes include mixture problems. You will be given a set of circumstances where one substance of a certain concentration must be added to a different quantity with an unlike concentration.

## Mixture Problem Formula

$$Q_1 C_1 + Q_2 C_2 = (Q_1 + Q_2)\ C_3$$

Q = Quantity or amount of substance

C = Concentration of each substance

Example: How many quarts of a juice that is 12% grape juice must be added to 6 quarts of juice that is 18% grapejuice to result in a mixture that is 15% grapejuice

Use Formula and translate

How many quarts of 12% grape juice

$(Q)(0.12)$

Must be added to

$+$

6 quarts of mixture of 18% GJ

$(6)(0.18)$

$=$

Mixture of 15%

$(Q+6)(0.15)$

Combine and Solve

$$(Q)(0.12) + 1.08 = (Q)(0.15) + 0.9$$

$$0.18 = 0.03q$$

$$6 = q$$

# Age Problems

$$\boxed{\text{Past}} \longleftrightarrow \text{subtract} \longleftrightarrow \boxed{\text{Now}} \longleftrightarrow \text{add} \longleftrightarrow \boxed{\text{Future}}$$

| 9 years ago | Present | 8 years from Now |
|---|---|---|
| $x-9$ | $x$ | $x+8$ |
| $x$ | $x+9$ | $(x+9)+8=x+17$ |
| $(x-8)-9=x-17$ | $x-8$ | $x$ |

Example: A father is 4 times as old as his son. In 8 years their ages will total 56 years. Find their current ages.

Solution:

| | Now | 8 years from Now |
|---|---|---|
| Father | $4x$ | $4x+8$ |
| Son | $x$ | $x+8$ |

$$56=(4x+8)+(x+8)$$ Translation from Problem

$$56 = 5x+16$$
$$40 = 5x$$
$$8 = x$$

Son $= 8$

Father $= 4(8) = 32$

Example: 5 years ago, the sum of the age of a mother and daughter was 34 years. 3 years from now, the mother will be 5 years greater than twice as old as her daughter will be. Find present ages.

Set up:

| | 5yrs ago | Now | in 3 yrs |
|---|---|---|---|
| Mother | $x$ | $x+5$ | $x+8$ |
| daughter | $34-x$ | $39-x$ | $42-x$ |

# Age Problems (cont.)

Translate:

| | |
|---|---|
| ① 3 years from now the mother will be | ① $x + 8 =$ |
| ② 5 years greater than | ② $5 +$ |
| ③ Twice as old as her daughter will be | ③ $2(42 - x)$ |

Put it all together in equation form:

$$\overset{①}{x+8 =} \quad \overset{②}{5+} \quad \overset{③}{2(42-x)}$$

Solve: $3x = 81$

$x = 27$

Present → $x+5$ → 32 mother

→ $39-x$ → 12 daughter

# Simple Probability and Basic Statistics

Probability = helps us predict the likelihood of chance occurrences.

Statistics = method of collecting, organizing, analyzing and interpreting data. Data is numerical information.

**Fundamental Counting Principle**

- A way to find the total number of possibilities
- If there are P ways one event can happen and there are Q ways another event can happen, then there are P * Q ways that both can happen together.

Test Sample using Fundamental Counting Principle

- If you have four ties, six shirts and 3 pairs of pants, how many different outfits can you make

4 ties * 6 shirts * 3 pants = 72

Permutations are ordered arrangements of things when:

→ No item is used more than once.

→ the ORDER of the arrangement is important and makes a difference.

Factorials are used for permutations.

→ $n! = n(n-1)(n-2) \cdots (3)(2)(1)$

→ $0!$ by definition $= 1$

$6! = 6 * 5 * 4 * 3 * 2 * 1 = 720$

There are 720 ways 6 books can be arranged on a shelf.

# Permutations of N things taken R at a Time.

- Sometimes you have a large set of items, but you can only take a few of them at a time for an arrangement.
- A good example is a baseball team. You have 13 players, but can only play 9 at a time. How many different batting orders can you make?

Formula for Permutations $_nP_r = \dfrac{n!}{(n-r)!}$

N → Total Number of items

r → how many Taken at a time.

$$_{13}P_9 = \frac{13!}{(13-9)!} = \frac{13!}{4!} = 259{,}459{,}200$$

Note: factorial growth is greater than exponential.

---

In a 52 card deck there are 52! different ways to arrange the cards after shuffling

$52! = 8.07 \times 10^{67}$ This number exceeds all the atoms on earth.

---

Example Problem: 5 children are standing in a line 2 are boys and 3 are girls. If the first child is a girl, how many different arrangements of the 5 children are possible?

Solution: Since there are 3 girls, there are 3 different ways to start. After the first girl, there are 4! ways to arrange.

$4! + 4! + 4! = 24 + 24 + 24 = 72$

# Combinations vs. Permutations

With permutations, the order of the elements <u>is</u> included in the total count of possibilities

With combinations, the different order or arrangements are **NOT** included.

For example: 3 letters taken 2 at a time

Permutation $_nP_r = {_3P_2} = \frac{3!}{(3-2)!} = \frac{3!}{1!} = 6$

| | | |
|---|---|---|
| AB | BA | CA |
| AC | BC | CB |

With combinations you don't care what order of the elements. You only care what elements are in each combination

AB would be the same as BA for a combination because both A and B are combined in each.

So there are only 3 combinations of 3 letters taken 2 at a time.

① AB = BA
② AC = CA
③ BC = CB

$$\boxed{\text{Combination Formula} \quad {_nC_r} = \frac{n!}{(n-r)!\,r!}}$$

Sample Problem $_3C_2 = \frac{3!}{(3-2)!\,2!} = \frac{6}{2} = 3$

# Probability and Data Collection

$$\text{Probability} = \frac{\text{frequency of what you want}}{\text{set of all possible outcomes}}$$

$$= \frac{\text{favorable outcomes}}{\text{all possible outcomes}}$$

$$= \frac{\text{Chance of specific outcome}}{\text{Total number of possible outcomes}}$$

= A number between 0 and 1 inclusive

If probabality equals 0, an event will NEVER happen.
If probability equals 1, an event will ALWAYS happen.

If a and b are mutually exclusive events, the probability that a or b will occur is equal to the sum of their individual probabilities.

Example: Event A → Coin flip $\frac{1}{2}$ H $\frac{1}{2}$ T
Event B → Picking a specific suit of cards from deck $\frac{1}{4}$

Chance of getting a head **OR** a spade.

$$\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4} = 75\%$$

Example: What is the chance of rolling a 7 or a 12 with one Roll of Two dice?

Chance of getting 7 + Chance of getting a 12

$$\frac{1}{6} + \frac{1}{36} = \frac{7}{36} = 19.\bar{4}\%$$

The probability of two mutually exclusive events both occurring is the product of their individual probabilities happening separately.

Example 1: What are the odds of getting both a head from a coin flip and a spade from a random pick of a deck of cards?

odds of getting heads → $\frac{1}{2}$

odds of getting a spade → $\frac{1}{4}$

$$\frac{1}{2} * \frac{1}{4} = \frac{1}{8} = 12.5\%$$

Example 2: What are the odds of getting a girl on your first birth and a girl on your second birth?

$$\frac{1}{2} * \frac{1}{2} = \frac{1}{4} = 25\%$$

Remember: If the words, "Either .... Or" are used Add the individual probability

If the words "BOTH ... AND" are used, MULTIPLY the odds.

# Averages

Averages are always on the test. Sometimes the question will ask straight out what the average is of a set of numbers. Most of time they will ask to determine what the final entry is needed to get a final average

Remember Average Formula →

$$\text{Average} = \frac{\text{Sum of Entries}}{\text{Total number of Entries}}$$

$$= \frac{x_1 + x_2 + x_3 + \ldots\ldots x_n}{n}$$

If all the entries are the same, then that number repeated is the average.

Most average problems have this set-up → what amount do you need for an unknown to be in order for the entire average to be a certain number

Example: To get an A, you need a 93 average or better on 5 tests. You scored 90, 89, 91 & 95. Can You still get an A?

Shortcut → You need 93 x 5 or 465 Points. You have 365 now. You need a perfect 100 To get an A.

If a series of numbers is evenly spaced, the average of the can easily be found by adding the first and last Number then dividing that sum by 2. For example,

3, 6, 9, 12, 15 All Numbers are Evenly SPACED

Average $\frac{3+15}{2} = \frac{18}{2} = 9$

Some ACT problems look for a missing element given the average sought.

Example: You want a 92 average for 5 tests. You have scores of 85, 96, 88 and 97 so far. What must you score on the fifth test to reach your goal?

Solution: $92 = \frac{85+96+88+97+x}{5}$

$460 = 366 + x$

$94 = x$

## Other important definitions

Median → The value that falls in the middle when a series of numbers is ranked from lowest to highest. If there is an even set of numbers, take the average of the two middle numbers

MODE → Value that appears the <u>MOST</u> often in a series of numbers.

# Weighted Averages

For some averages, greater "Weight" is given to each element individually. The frequency or likelihood of the elements is considered.

Example 1: 3 books cost $6 each. 4 books cost $10 each and 5 books cost $12 each. What is the average cost per book?

Answer: It's not ($6 + $10 + $12) ÷ 3 because their are different amounts of books and each book has a different price.

Factoring "Weights" the equation looks like

$$\frac{3(\$6) + 4(\$10) + 5(\$12)}{3+4+5}.$$

$$\frac{18+40+60}{12} = \frac{118}{12} = \$9.84$$

To find weighted averages, multiply each number in the set by the number of times it appears in the set. Finally, add them all up and divide by the total number of items.

Example 2 In a group of ten students, 7 are 13 and 3 are 17. What is the average age of the students?

ANSWER $\left[(7 * 13) + (3 * 17)\right] \div 10 = 14.2$

# Weighted Averages

Example 3: The final exam is worth Twice the weight of the other Two mid-terms. Mid-term one score is 75, mid-Term 2 score is 85 and Final score is 90. What is the average for the class.

ANSWER: $$\frac{75 + 85 + 2(90)}{4} = 85$$

You divide by 4 and not 3 because final is worth 2 Tests

Important Note about Averages

"You can't average averages"

If your batting average is 0.250 the first half of the season and your average for the second half is 0.350 your average for the season is not $(0.250 + 0.350) \div 2$.

You may have had 100 at bats the first half of the season and only 10 at bats the second half.

You have to take your total hits and divide by your TOTAL at bats to find out your average for the season.

# Arithmetic and Geometric Sequences

Arithmetic Sequences have a common difference between the numbers in the Series.

Arithmetic sequences increase or decrease by addition or subtraction of common differenc

## SUM of Arithmetic Series Formula

$$\text{Sum of Series} = \left(\frac{a_1 + a_n}{2}\right) N$$

$a_1$ = 1st term  $a_n$ = last term  $N$ = Number of terms

In words, the sum of an arithmetic series equals the product of the average of the first and last terms and the number of terms

For example: −5, −2, 1, 4, 7, 10, 13
What is the sum of this series?

Solution: First, Verify its an arithmetic series by determining a common difference.

$13-10=3$  $4-1=3$  $-2-(-5)=3$

3 → common difference

To find sum, add −5 and 13 then divide by 2 $(-5+13)\div 2 = 8 \div 2 = 4$

Multiply 4 by # of terms $= 4 \times 7 = 28$

Sum of Series = 28

# Formula to find a specific element of an arithmetic SERIES

$$A_n = A_1 + (N-1)D$$

Terms defined $A_n$ = The specific element in series. $A_4$ is the fourth term in series. $A_n$ can also be used for the last term in series.

$A_1$ = First term in series

$N$ = Number of terms in series

$D$ = the common difference

---

Example: Find the 8th term in the arithmetic series

$0.7, 1.1, 1.5, \ldots$

First → find common difference

$1.1 - 0.7 = 0.4$ and $1.5 - 1.1 = 0.4$

$A_8 = 0.7 + (8-1)0.4$

$A_8 = 0.7 + (7)(0.4)$

$A_8 = 0.7 + 2.8$

$A_8 = 3.5$

# Geometric Series

A sequence is geometric if there is a pattern of each term produced by multiplying or dividing the preceding term by a common Ratio. R = common ratio.

Series creation: $A_1 = A$ A = 1st term

$A_2 = AR$

$A_3 = AR^2$

$A_4 = AR^3$

Therefore, to find the nth term of a geometric series, use the formula

$$A_n = AR^{(N-1)}$$

Where N = the last term or a particular numbered element in series.

Example: Find the 6th term in the series

$16, -4, 1, \ldots$

First, find the common ratio by setting up proportion.

$$\frac{16}{-4} = \frac{1}{R} \Rightarrow 16R = -4$$

$$R = \frac{-4}{16} = \frac{-1}{4} = \text{Common Ratio}$$

Apply formula $A_6 = 16\left(\frac{-1}{4}\right)^{(6-1)}$

$$= 4^2\left(\frac{-1}{4}\right)^5$$

$$= 4^2 * \frac{-1}{4^5} = -\frac{1}{64}$$

Formula to find the first N terms of a geometric series

$$\text{Sum of } N \text{ terms} = \frac{A_1 - A_1 R^N}{1 - R}$$

For example, use our previous series 16, −4, 1 To find the sum of the first three terms.

$$\text{Sum of } 1^{st} \text{ three terms} = 16 + (-4) + 1 = 13$$

$$\text{Using formula} \Rightarrow \frac{16 - 16\left(\frac{-1}{4}\right)^3}{1 - \left(\frac{-1}{4}\right)} =$$

$$\frac{16 - \left[4^2\left(\frac{-1}{4^3}\right]\right.}{\frac{5}{4}} = \frac{16 - \left(-\frac{1}{4}\right)}{\frac{5}{4}} = \frac{\frac{65}{4}}{\frac{5}{4}} = \frac{65}{5} = 13$$

---

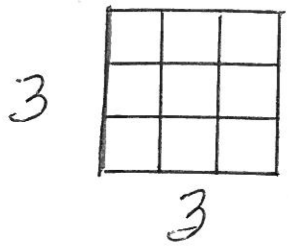
Triangular numbers formula

$$\frac{n(n-1)}{2}$$

Nth term

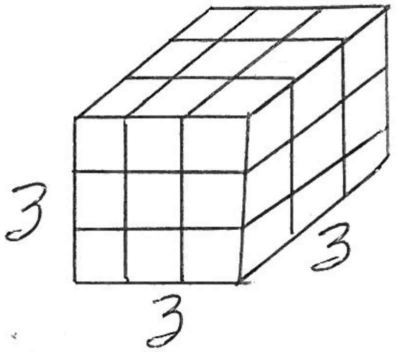
# Exponents and Roots

Just like multiplication is repeated addition, exponents are repeated multiplicatio



$3 * 3 = 3^2 = 9$

$3^2$ is read "3 to the 2nd Power or 3 square



$3 * 3 * 3 = 3^3 = 27$

$3^3$ is read "3 to the third power" or "3 cubed

The number being multiplied is called the base and the superscript is called the exponent. the exponent says how many times the base gets multiplied.

You should memorize all perfect squares up to 15 base.

example

If $a < b$, then $a^n < b^n$ $3^2 < 4^2$

Negative exponents represent the reciprocal of a positive power

$x^{-n} = \frac{1}{x^n}$ example $3^{-4} = \frac{1}{3^4}$

# Rules for Exponents

When adding expressions with variables, add the coefficients only

Example: $3x^2 + 5x^2 = 8x^2$

---

When multiplying powers with the same base, keep the base but add the exponents.

Rule: $x^n * x^m = x^{n+m}$ Ex. $x^2 * x^5 = x^{2+5} = x$

---

When dividing powers with same base, keep the base and subtract the exponents

Rule: $\dfrac{x^n}{x^m} = x^n \div x^m = x^{n-m}$

---

When raising a fraction to an exponential power, you can apply the exponent to both numerator and denominator.

Rule: $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n} = \left(\dfrac{x}{y}\right)^n$

---

When you raise a power to another power you multiply the power.

$(x^n)^z = x^{nz}$ Example: $(x^4)^5 = x^{4*5} = x^{20}$

---

When factors are raised to a power, you can distribute the exponent to each factor.

$(ab)^r = a^r b^r$ $(a^n b^s)^r = (a^n)^r (b^s)^r = a^{rn} b^{sr}$

# More rules for Exponents

Any number to the power of one equals the base.

Rule: $n^1 = n$ Example. $5^1 = 5$

---

Any base to the power of zero equals 1

Rule: $n^0 = 1$ Example: $5^0 = 1$

---

When raising a fraction to a power greater than one, the result gets smaller

Example: $\left(\frac{3}{4}\right)^3 = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64} < \frac{3}{4}$

---

When a fraction is raised to a negative power, flip the numerator and denominator and make the exponent positive

Rule: $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$ $\left(\frac{x}{y}\right)^{-1} = \frac{y}{x}$

---

When raising a negative base to any odd power, the number gets smaller

Example: $-3^3 = (-3)(-3)(-3) = -27$

$-27 < -3$

---

Negative Exponents are reciprocals

$$\frac{a^{-1}}{b^{-1}} = \frac{b^1}{a^1} = \frac{b}{a}$$

# Scientific Notation (SN)

SN is a method to express extremely small or extremely large numbers in a confined space.

To convert, you move the decimal point to the right of the leading place value. You then multiply by a base ten number raised to the exponential power equal to the number of place value shifts in the original number.

For large numbers, you will shift the decimal point from <u>RIGHT TO LEFT</u> and the exponential value will be <u>POSITIVE</u>

EXAMPLES: $10 = 1.0 \times 10^1$

$100 = 1.0 \times 10^2$

$5{,}250{,}000 = 5.25 \times 10^6$

For small numbers, you will shift the decimal point from <u>LEFT TO RIGHT</u> and the exponential power will be <u>NEGATIVE.</u>

Examples: $\frac{1}{10} = 0.1 = 1.0 \times 10^{-1}$

$\frac{1}{1000} = 0.001 = 1.0 \times 10^{-3}$

$0.0000000183 = 1.83 \times 10^{-8}$

Notice the number of place value shifts equals the exponential value.

# Scientific Notation (cont)

When converting a number in SN to a regular number, if exponent is positive move decimal point to the **right** the same number of place values as the exponent value.

If the exponent is negative, move the decimal point to the **Left**

When multiplying numbers in SN use addition rule of multiplying numbers with exponents.

- Example: $(5 \times 10^6)(3 \times 10^4) = (5 \times 3)(10^6 \times 10^4)$

$$= 15 \times 10^{10}$$

$$= 1.5 \times 10^{11}$$

When dividing numbers in SN apply the subtraction rule.

$$(5 \times 10^{-5}) \div (4 \times 10^4) = \frac{5 \times 10^{-5}}{4 \times 10^4}$$

$$= 1.25 \times 10^{(-5-4)}$$

$$= 1.25 \times 10^{-9}$$

# Geometric Sequences with ONly Variables.

Sometimes you'll be given a geometric sequence with only variables and you'll be asked to find the next term in the series

Example: What is the 4th term of the geometric sequence

$2xz, 2x^2yz, 2x^3y^2z, \ldots$

Solution: Make a proportion

$$\frac{2xz}{2x^2yz} = \frac{2x^3y^2z}{\text{Next Term in Sequence}}$$

$$\frac{(4^{\text{th}}\text{ Term})\,2xz}{2xz} = \frac{(2x^2yz)(2x^3y^2z)}{2xz}$$

$4^{\text{th}}$ term $= 2x^4y^3z$

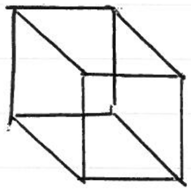
# Radical Expressions

A radicand is an expression contained within the symbol $\sqrt{\phantom{x}}$

These expressions are called "roots"

Square roots find the same factor for a number to the second power. Square roots can also find the sides of a square given the area.

$\boxed{64}$ $s=8$ If area $=64$, then $\sqrt{64}=8$ because $8*8=64$. Same factor is 8.



$s=4$ If Volume $=64$, then $\sqrt[3]{64}$ equals 4 because $4*4*4=4^3=64$

## Rules for Radicals

$\sqrt{x}+\sqrt{x}=2\sqrt{x}$ Ex: $2\sqrt{5}+3\sqrt{5}=5\sqrt{5}$

Product Rule: $\sqrt{x}\cdot\sqrt{x}=\sqrt{x\cdot x}=\sqrt{x^2}=x$

$$\sqrt{xy}=\sqrt{x}*\sqrt{y}=\sqrt{xy}$$

Example: $\sqrt{12}\sqrt{3}=\sqrt{12\cdot 3}=\sqrt{36}=6$

Quotient Rule:

$$\sqrt{\frac{x}{y}}=\frac{\sqrt{x}}{\sqrt{y}}$$ Ex: $\sqrt{\frac{3}{16}}=\frac{\sqrt{3}}{\sqrt{16}}=\frac{\sqrt{3}}{4}$

Square root of a fraction less than 1 is larger than original fraction

Example: $\sqrt{\frac{1}{4}}=\frac{\sqrt{1}}{\sqrt{4}}=\frac{1}{2}$

When taking square roots of a number $> \underline{1}$,
the root is smaller

Need to know how to approximate square roots
And cube roots

Example: $\sqrt{120}$ is close to $\sqrt{121} = 11$
$\sqrt{2}$ is close to 1.4
$\sqrt{3}$ is close to 1.7

Know how to simplify radicals.

ex. $\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$

$x^{1/N}$ = the $n^{\underline{th}}$ root of $x$

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m = \sqrt[n]{x^m}$$

Example:
$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \sqrt[3]{8^2} = 4$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\ \sqrt[n]{y}$$

$$\sqrt{xy} = \sqrt{x}\ \sqrt{y}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Cube roots of Negative Numbers are real, but Square roots are Not

$$(-1)^n \rightarrow \begin{cases} +1 \text{ if } n \text{ is even} \\ -1 \text{ if } n \text{ is odd} \end{cases}$$

$\sqrt{y-4} = 4$ square both sides $\left(\sqrt{y-4}\right)^2 = (4)^2$

$y - 4 = 16$
$y = 20$

If $\sqrt[b]{a} = c$ then $c^b = a$

# Rationalize Denominators

The denominator in a fraction is the divisor. $\frac{3}{4} = 4\overline{)3}$.

Divisors must be integers. You can't divide into fractional lots.

Radicals can produce irrational numbers. Sine irrational numbers are not integers, they can't be divisors or denominators.

To get rid of radicals in denominator, multiply the fraction by the denominator in unit form.

Example $\frac{1}{\sqrt{2}}$ $\sqrt{2}$ in unit form is $\frac{\sqrt{2}}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2 \cdot 2}} = \frac{\sqrt{2}}{2}$$

---

## Imaginary Numbers

Since the square of a negative number is positive, then it can't be negative. The square of a real number can't be negative. Therefore, the square root of a negative number is not real; it's imaginary.

$i = \sqrt{-1}$

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$i^5 = i$

$i^6 = -1$

$i^7 = -i$

$i^8 = 1$

The sum of a real number an an imaginary number is called a Complex Number

$3 + 2i$ is a complex number

$$\begin{aligned}(1+3i)(5-2i) &= 5-2i+15i-6i^2 \\ &= 5+13i-6(-1) \\ &= 5+13i+6 \\ &= 11+13i\end{aligned}$$

# LOGARITHMS

By definition, if $b^y = x$, then $y = \log_b x$

As you can see, logarithms ARE the exponent

$\log_b x$ is translated as "b to what power equals x?"

We use b in the sample because it means base

Logarithms are the inverse of an exponential function because the base and the exponent get inverted.

If $2^3 = 8$, then $\log_2 8 = 3$

The base remains the same. The exponent and the product switch places. Just like how exponents exist in terms of a base, logs always are in terms of some base.

Logs that don't indicate a base are called common logarithms. Their base is 10

$\log x$ is the same as $\log_{10} x$

Natural logarithms have e as a base

$\ln 6 = \log_e 6$ $\quad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

## Change of base formula

If $\log_b x$ then $\frac{\log_{10} x}{\log_{10} b}$ or $\frac{\ln x}{\ln b}$

# Rules for Logarithms

Since logs and exponents are inverse operations, the rules for each are closely related

| Rule for exponent | Corresponding Rule for logarithm |
|---|---|
| $b^x b^y = b^{(x+y)}$ | $\log_b XY = \log_b X + \log_b Y$ |
| $\frac{b^x}{b^y} = b^{(x-y)}$ | $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$ |
| $(b^x)^y = b^{xy}$ | $y \log_b x = \log_b x^y$ |
| $b^1 = b$ | $\log_b b = 1$ |
| $b^0 = 1$ | $\log_b 1 = 0$ |
| $b^x = b^x$ and If $b^u = b^w$, then $u = w$ | $\log_b b^x = x$ |
| $b^y = x$ | $b^{\log_b x} = x$ |

## Example One

If $x = \log_4 \sqrt{64}$, what is value of $x$?

Think 4 to what power $= \sqrt{64}$

$4^x = \sqrt{64}$

$(4^x)^2 = (\sqrt{64})^2$

$4^{2x} = 64$

$4^{2x} = 4^3$

$2x = 3$

$x = \frac{3}{2}$

## Example Two

If $\log_7 x^3 = 9$, what is $x$?

$\log_7 x^3 = 3 \log_7 x = 9$

$\frac{3 \log_7 x}{3} = \frac{9}{3}$

$\log_7 x = 3$

$x = 7^3 = 343$

# Algebra

Algebra could be defined as "using what you know to find out what you don't"

Mathematics is a system. As with any system, math is comprised of parts working together that creates patterns. Once you know the "rules of the road," you will use these patterns to solve problems.

Mathematics is also a language. When you do algebra, you translate from English to algebra. Here is a list of the common phrases:

| English Words | Math symbols |
|---|---|
| is, as, was, has, costs | $=$ |
| More than, plus, together, total, sum of, EArNS | $+$ |
| spends, loses<br>Less than, difference | $-$ |
| "of", the product of, times | $()()$ $\times$ or $*$ or $\cdot$ |
| quotient, broken into, cost per unit | $\div$ $\frac{a}{b}$ |
| X percent, what percent | $\frac{x}{100}$ |
| a certain number, what Number, an unknown | letters used for any Number<br>a, x, y, z, etc. |

# Translation from English to Algebra (cont.)

You need a separate algebraic sentence for every unknown

| English | Algebra |
|---|---|
| $x$ and $y$<br>$x$ combined with $y$<br>$x$ more than $y$<br>$y$ more than $x$<br>$x$ older than $y$<br>$x$ increased by $y$ | $x + y$<br>or<br>$y + x$ |
| the difference between $x$ and $y$<br>$y$ less than $x$<br>The decrease from $x$ to $y$ | $x - y$ |
| the difference between $y$ and $x$<br>$x$ less than $y$<br>the decrease from $x$ to $y$ | $y - x$ |
| the product of $x$ and $y$ | $xy$ or $x * y$ |
| $m$ in terms of the product of $x$ and $y$ | "in terms of" means to solve for that variable<br>$m = xy$ |
| the square of $x$ | $x^2$ |
| the cube of $x$ | $x^3$ |

# More translations

| English | Algebra |
|---|---|
| x is greater than y | $x > y$ or $y < x$ |
| x is less than y | $x < y$ or $y > x$ |
| y years ago | $-y$ |
| y years from now | $+y$ |
| Percent increase from x to y | $\left(\frac{y-x}{x}\right)100$ |
| Percent decrease from x to y | $\left(\frac{x-y}{x}\right)100$ |

When working on problems, most of the time its best to solve the problem without looking at the answers. then, match up your answer with one of the choices.

However, when the question has variables and the answers have real numbers, use the answers as a "which of the following"

## USING ALGEBRA

After translating, you need to work with the algebraic system. To work in any system you need to know the rules of the road. You use all the basic rules of arithmetic, all the basic rules of numbers, and special rules we will discuss now.

When combining and simplifying algebraic sentences, only "like" terms can be combined. "Like" terms have the same level of exponent and product combination. For example,

$$x^3y + x^3y \qquad\qquad x^2y^2 + x^3y^2$$

These are like terms
Can be combined to $2x^3y$

Unlike terms
Can't be added.

When multiplying expressions with variables and coefficients, follow the same rules as any number. Example:

$$\frac{1}{2}x * 3x = \frac{1}{2} \cdot 3 \cdot x \cdot x = \frac{3}{2}x^2$$

If bases are different, just show them multiplied

$$3x \cdot 3y = 3 \cdot 3 \cdot x \cdot y = 9xy$$

When multiplying <u>monomials</u>, just multiply the coefficients and variables separately like the above operation.

When multiplying <u>binomials</u> such as $(x+7)$ and $(x+3)$, you have to multiply each term with the other terms.

Use FOIL method when multiplying binomials. or you can split up and use distributive property $(x+7)(x+3) = x(x+3) + 7(x+3) =$
$x^2 + 3x + 7x + 21 = x^2 + 10x + 21$

# FOIL METHOD

F.O.I.L. Stands for First, Outer, Inner, Last

$$(A + B)(C + D) = AC + AD + BC + BD$$

**F**irst → A times C
**O**uter → A times D
**I**nner → B times C
**L**ast → B times D

Equations that look like this are called **QUADRATICS** because 4 terms are multiplied

You need to memorize these common quadratics

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (a+b)(a-b) = a^2 - b^2$$

→ This is called "The difference of two squares."

Equations in the form of $Ax^2 + Bx + C = 0$ can be factored using the foil method. However, sometimes it's not very easy.

When there are rational roots (solutions for x) you can use the quadratic formula

In equations of form $Ax^2 + Bx + C = 0$

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

For the A.C.T., use this rarely. Try to factor first.

# Guidelines for factoring equations in form $Ax^2+Bx+c=0$

To easily factor quadratics w/o using the cumbersome quadratic formula, follow these Rules:

1) When the second sign is "+," then the signs of the binomials are either both "+" or both "−" Example: $x^2-9x+14=(x-7)(x-2)$
$x^2+9x+14=(x+7)(x+2)$

2) If the first sign is "−" and the second sign is "+," then both binomials have the "−" sign

3) If the second sign is "−," then the two binomials have different signs. Example:

$$(2x^2-5x-12)=(2x+3)(x-4)$$

4) To find the C term, discover what two numbers when multiplied produce that number. Then find which of those factors produce the B term when added. Example:

$$x^2-x-6=(x+2)(x-3) \text{ because}$$

When Two Numbers are multiplied they produce −6

$$\begin{cases} -6 * 1 = -6 \\ -1 * 6 = -6 \\ 3 * (-2) = -6 \\ 2 \times (-3) = -6 \end{cases}$$

And when Two Numbers are added they produce −1 because $-x = -1x$

$$\begin{cases} -6 + 1 = (-5) \neq -1 \\ -1 + 6 = 5 \neq -1 \\ 3 + (-2) = 1 \neq -1 \\ 2 + (-3) = (-1) = (-1) \end{cases}$$

$$x(y \pm z) \pm w(y \pm z) = (x \pm w)(y \pm z)$$

## Summary on factoring $Ax^2+Bx+c=0$

If $Ax^2+Bx+c$, then $(\ +\ )(\ +\ )$

If $Ax^2-Bx+c$, then $(\ -\ )(\ -\ )$

$\boxed{\begin{array}{l}\text{If } Ax^2-Bx-c\\ \text{If } Ax^2+Bc-C\end{array}}$ then $(\ +\ )(\ -\ )$ or $(\ -\ )(\ +\ )$

Find Two Numbers when multiplied produce C and the same Two Numbers will produce B when Added.

Example: $x^2+4x-21=0$

If 7 and $(-3)$ are multiplied → $-21$

If 7 and $(-3)$ are added → 4

Therefore: $x^2+4x-21=(x+7)(x-3)=0$

Roots are $\underline{\underline{-7}}$ and $\underline{\underline{3}}$

## Sum and Product Rule for Roots

$$\boxed{\text{Sum of Roots} = \frac{-B}{A} \qquad \text{Product of Roots} = \frac{C}{A}}$$

From our example: $x^2+4x-21$

Sum → $\frac{-B}{A} = \frac{-4}{1} = (-4)$ Roots were $-7$ and $3$
Sum of $-7$ and $3$ is $-4$

Product $\frac{C}{A} = \frac{-21}{1} = (-21)$ Roots were $-7$ and $3$

$-7 * 3 = (-21)$

# Factoring Algebraic Fractions

Two basic ways to simplify
1) Factor
2) Combining like terms

Factoring means finding common factors and dividing them out. Examples

$$6x + 8y = 2(3x + 4y)$$ 2 is a common factor of 6 and 8

$$\frac{2x+y}{x} = \frac{2x}{x} + \frac{y}{x} = 2 + \frac{y}{x}$$

$$\frac{x^2 - x - 12}{x^2 - 9} = \frac{(x-4)\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = \frac{x-4}{x-3}$$

Combining like terms

– "like terms" are those terms that have the same variables with the same exponent value. All constants or real numbers like 3, 4.2, $+\frac{3}{7}$ are like terms.

$$m^2 + m^2 = 2m^2 \text{ but } x^4 + x^2 \neq x^6$$

The only thing you can do with $x^4 + x^2$ is factor out the $x^2$. So $x^4 + x^2 = x^2(x^2+1)$

Sometimes you need to multiply out then combine like terms

$$\begin{aligned} x(x-4) - 4(4-x) &= x^2 - 4x - 16 + 4x \\ &= x^2 - 16 \\ &= (x+4)(x-4) \end{aligned}$$

# Additional info on Algebraic Factoring

1) Evaluating means substituting values for x then solving for x.

For example: Evaluate $-x - x$ when $x = -2$

Solution: Wherever you see x, substitute $-2$

$$-(-2) - (-2) = 2 + 2 = 4$$

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2) If $a * b = 0$, then either $a = 0$ or $b = 0$ or both.

Application: $(x-4)(3x+1) = 0$

$x - 4 = 0$ or $3x + 1 = 0$

$x = 4$ or $x = -\frac{1}{3}$

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3) When solving algebraic inequalities, remember to reverse the sign when dividing or multiplying by a negative number

$$-3x \geq 2$$

Dividing by -3

$$\frac{-3x}{-3} \leq \frac{2}{-3}$$

$$x \leq -\frac{2}{3}$$

# Coordinate Graphing or Analytic Geometry

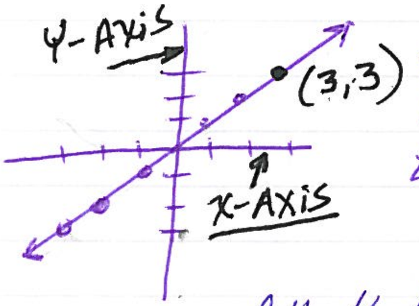
Linear equations are just what the name implies → they are the algebraic expressions of the geometric shape of a line

Algebraic sentences with two variables each with a power of 0 or 1 create straight lines on a coordinate plane.

The simplest possible line is $Y = X$ every other line is a variation of this line

Note

Connecting Two Points create one and only one line



Every point on this line satisfies the unique equation

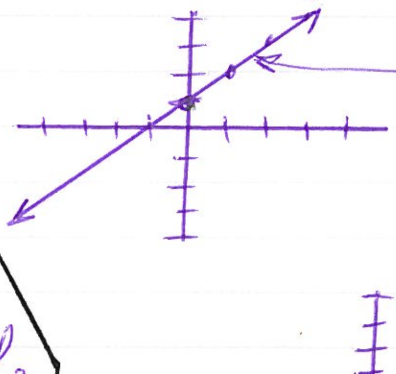
$$y = x$$

All Y Points are the same as all x points

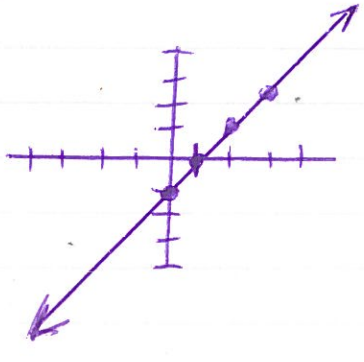
The equation $Y = x + 1$ creates a shift of one unit up from $y = x$.

Note:

Every unique geometric shape or position in the x:y plane has one unique equation



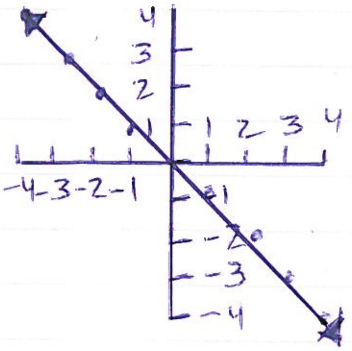
$y = x + 1$ All Y values Equal all x values plus 1



$y = x - 1$ is a shift down 1 from $y = x$

Watch what happen's when you inverse the x values in

$y = -x$



See how the line has reversed direction

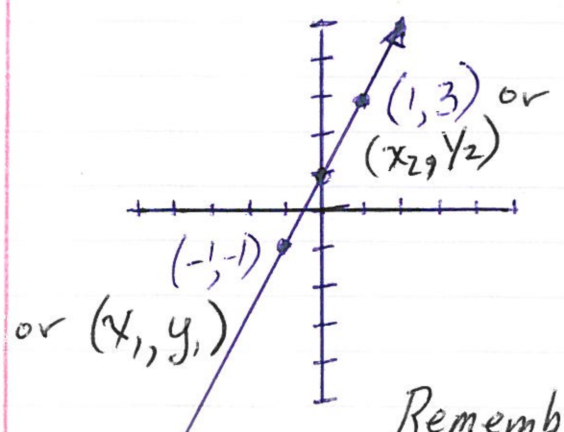
## Slope of Lines

Slope is simply a measurement. It's a measure of a line's Steepness. This measurement is the result of <u>COMPARING</u> the change of the line going vertical and the line going horizontal.

Since Slope is a Comparison, its A RATIO

Slope is RATIO of change in y values divided by change in x values.

$$\text{Slope} = m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$



To determine Slope when you're only given Two points, subtract the coordinates like this

$$m = \frac{\Delta y}{\Delta x} = \frac{3-(-1)}{1-(-1)} = \frac{4}{2} = 2$$

Remember: Two points make <u>ONE</u> and only one unique. line

# 3 ways to express a line and using Algebra to graph lines.

1) General Form $Ax + By = C$

This form is good for graphing lines using x and y intercepts.

Intercepts → the point where a line crosses the x or y axis.

When a line crosses the x-Axis, the y coordinate is always 0.

when a line crosses the Y-axis, the x coordinate is always 0.

Let's graph a line in General form $Ax+By=C$

Example: $3x + 4y = 24$

Knowing that when $x=0$, the line will cross the y-axis at one specific point. So, make $x=0$ and solve for y to find the Y-intercept.

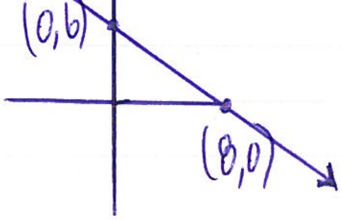
When $x=0$ then $y=6$ $3(0) + 4y = 24 \rightarrow 4y = 24 \rightarrow y=6$

Y-intercept point is (0, 6)

When $y=0$ then $x=8$ $3x + 4(0) = 24 \rightarrow 3x = 24 \rightarrow x = 8$

x-intercept point is (8,0)

Graph using y-intercept (0, 6) + x-intercept (8,0)



Two points create ONE unique line

# WAys to algebraicly express STraight lines

## 2) Slope - Intercept Form looks like:

$y = mx + b$ Where $m$ = Slope
$b$ = y-intercept

You can find slope and y-intercept from any line equation $Ax + By = C$, by using these formulas:

Slope $= m = \dfrac{-A}{B}$ y-intercept $= b = \dfrac{C}{B}$

Example: $3x + 4y = 24$

Long way:
Convert to
$y = mx + b$
form

$$\begin{array}{r} 3x + 4y = 24 \\ -3x \qquad -3x \\ \hline \end{array}$$

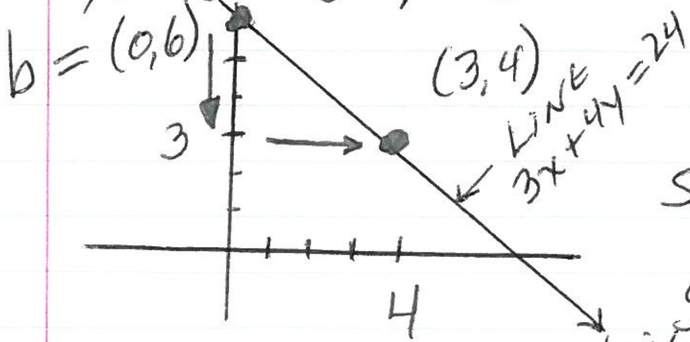
$$\frac{4y}{4} = \frac{-3x + 24}{4}$$

$$y = \frac{-3}{4}x + 6$$

Short way $M = \dfrac{-A}{B} = \dfrac{-3}{4}$ y-intercept $= \dfrac{C}{B}$

$b = \dfrac{24}{4} = 6$

ONCE you have slope and y-intercept you can graph the line

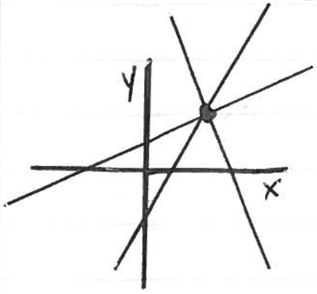


START with $b = 6$
POINT is (0,6) because $x = 0$ for the y-intercept
Slope $= m = \dfrac{-3}{4}$ means go down (negative) 3 and right (POSITIVE) 4. 2nd POINT

# Ways to express straight lines (cont.)

## 3) Point - Slope Form

If you have the coordinates for one point and the slope of a line passing thru that point, you can create an equation



There are an infinite number of lines going thru one point, but given one slope, one and only one line can be drawn thru it.

The general form for point-slope is:

$$(y_2 - y_1) = m(x_2 - x_1)$$

This form is really just a variation of the equation for the definition of slope

$$M = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

Solving for $y_2 - y_1$ we get

$$(x_2 - x_1) \cdot m = \frac{(y_2 - y_1)}{\cancel{(x_2 - x_1)}} \cdot \cancel{(x_2 - x_1)}$$

Commutative property

$$m(x_2 - x_1) = (y_2 - y_1)$$

Switching sides does not change anything

$$(y_2 - y_1) = m(x_2 - x_1)$$

Point - slope Form

Finding equation of line given one point and one slope example.

Slope $= \frac{1}{2}$  Point $(3, 2)$

Plug into formula: $(y_2 - y_1) = m(x_2 - x_1)$

$y - 2 = \frac{1}{2}(x - 3)$ or $2 - y = \frac{1}{2}(3 - x)$

$y = \frac{x}{2} - \frac{3}{2} + 2$

$y = \frac{1}{2}x + \frac{1}{2}$

$2 - y = \frac{3}{2} - \frac{x}{2}$

$\frac{x}{2} + 2 + \left(\frac{-3}{2}\right) = y$

$\frac{1}{2}x + \frac{1}{2} = y = \frac{1}{2}x + \frac{1}{2}$

## Systems of Equations

Lines in a plane are parallel, perpendicular or just simply intersect in a random manner.

When lines are perpendicular, their slopes are <u>opposite reciprocals</u>

If slope $line_1$ is $\frac{y}{x}$ and slope of $line_2$ is $-\frac{x}{y}$, then the lines meet at 90°

When lines are parallel, slopes are equal

If slope $line_1 = \frac{y}{x}$ and Slope of $line_2$ Equals $\frac{y}{x}$ then the lines are equal.

# Systems of Equations (cont.)

If slopes of any two lines are unequal, then the lines intersect.

This point of intersection is the solution to a systems of equations.

Example:

$$\left.\begin{array}{l} y = x - 1 \\ y = 2x + 1 \end{array}\right\}$$

as you can see from slope-intercept form, these lines have different slopes

Since the lines have different slopes, the lines will intersect and this point of intersection will be the solution.

Solution: Substitute one value of y into the second equation and solve for x.

Since y is the same as x − 1, substitute

$$\begin{array}{rcl} x - 1 & = & 2x + 1 \\ -(x-1) & & -(x-1) \\ \hline 0 & = & x + 2 \\ -2 & & -2 \\ \hline -2 & = & x \end{array}$$

$$X = -2$$

Now that we know $x = -2$, we substitute back into first equation and solve for y.

$$y = (-2) - 1$$

$$y = -3$$

The point of intersection is $(-2, -3)$

# Summary and additional information for Analytic Geometry

A line is a set of points represented in the general form

$$Ax + By = C$$

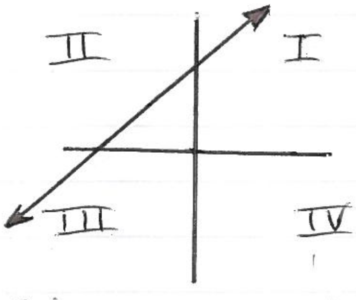
Y-intersept occurs when $x = 0$
The y-intersept point is $\left(0, \frac{C}{B}\right)$

X-intersept occurs when $y = 0$
The x-intersept point is $\left(\frac{C}{A}, 0\right)$

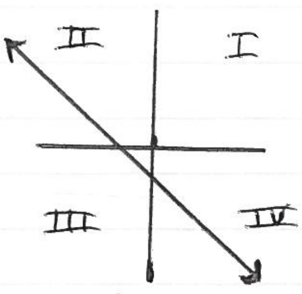
When two lines intersect → one solution
When two lines are parallel → No solution
When two lines equal → infinite solutions.

Mid-Point formula → $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

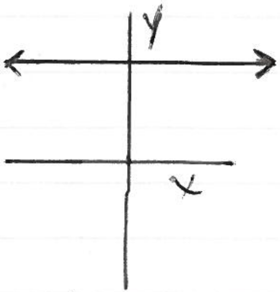
Distance Formula $= \sqrt{(x_2 - x_1) + (y_2 - y_1)}$



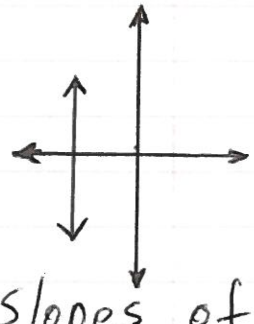
Slope is positive if goes thru both I and III



slope is Negative Both II + IV



Horizontal Lines have 0 slope



Slopes of vertical lines dont exist

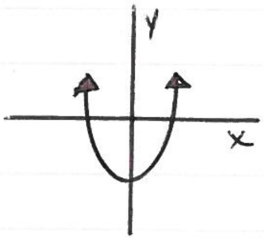
Distance on a Number line

$D = |B - A| = |A - B|$

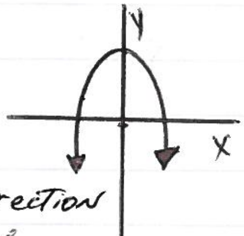
# Equations in the form $Ax^2 + Bx + C = 0$

Notice how one x variable has a power other than 0 or 1. Therefore, this equation will <u>NOT</u> produce a straight line.

Equations in the form $Ax^2 + Bx + C = 0$ will produce a parabola and is called a <u>quadratic equation.</u>



When $A > 0$, then parabola opens UP



When $A < 0$, then parabola opens DOWN

Vertex = $(-b/2a, f(-b/2a))$

Vertex is point where line changes direction

$C$ = Y-intercept or where the graph crosses the x-axis.

$Ax^2 + Bx + C = 0$ is also expressed in form

$$y = a(x-h)^2 + k$$

again, if $a > 0$, graph opens up.
If $a < 0$, graph opens down
if $a = 0$, graph is horizontal line

Coordinate $(h, k)$ is the <u>Vertex</u> or the point where the parabola changes direction.

Vertex also equals $\left(-\frac{b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

# Functions

A function is a rule applied to a set of numbers where one and only one result occurs. The numbers that can be applied to the rule are called the <u>DOMAIN</u>. The resulting numbers produced when applying the rule is called the <u>RANGE</u>

Domain Set → x-values
RANGE SET → y-values

Functions appear like regular algebraic problems using x and y, but instead of y equaling something we replace y with $f(x)$

A good example is the slope-intercept form

$$y = mx + b \Rightarrow f(x) = mx + b$$

To be a true function m and b need to be replaced with real numbers.

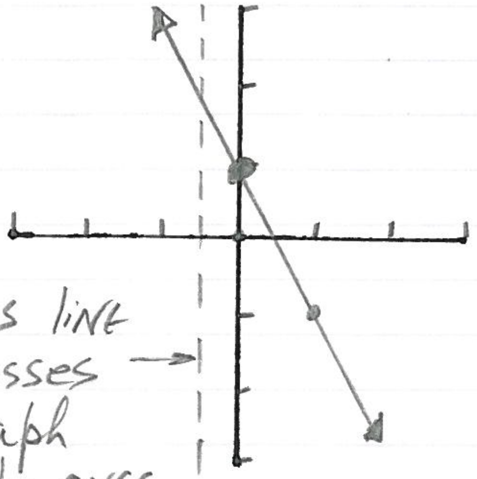
$$f(x) = -2x + 1$$

Graphs show coordinates of all the results when we input all the domain or the possible values of x. For this function, x could be any real number. However, some functions will not produce a result for some values of x. For example:

$$f(x) = \frac{5}{x+2}$$ x can't be $-2$ because $\frac{5}{0}$ would result.

# FUNCTIONS (CONT.)

For the function $f(x) = -2x + 1$, the graph will look like:



This line crosses graph only once

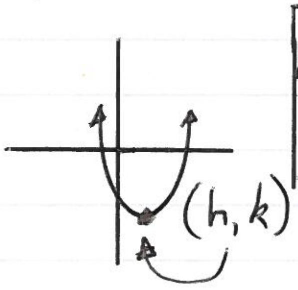
$f(x) = -2x + 1$ is a function because for every unique $x$ input, there will be one and only one output.

## Vertical line test

If you can draw any vertical line from any $x$-value and that line intersects the graphed line at one and only one point, then its a true function.

graph of $f(x) = a(x-h)^2 + k$

If $a > 0$



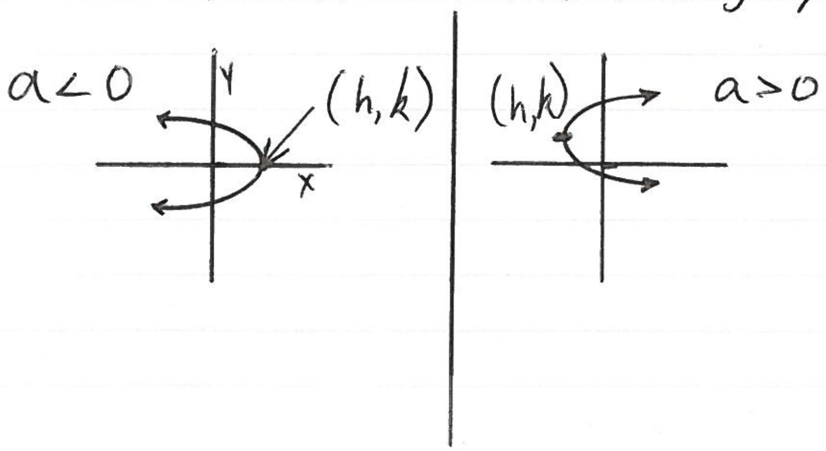
is the vertex

as a gets larger, the graph appears steeper.

as a gets smaller graph get flatter

If equation is in the form

$x = a(y-k)^2 + h$ then this is NOT a function since the graphs looks like these



Since these graphs do NOT pass the vertical line test, they are NOT functions

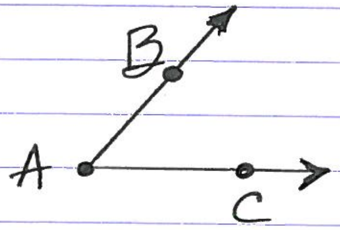
# Plane Geometry

## 14 Questions or about 25% of Exam

When a diagram is not provided, draw one
A line is an infinite set of points that
run straight
Line segment has two distinct end points
The midpoint of a line segment is 1/2 way
between each end point.

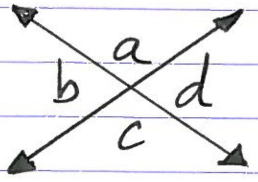
A RAY is a line with only one endpoint.
- A Ray does not have a midpoint

An ANGLE is two rays connected by
a common endpoint.



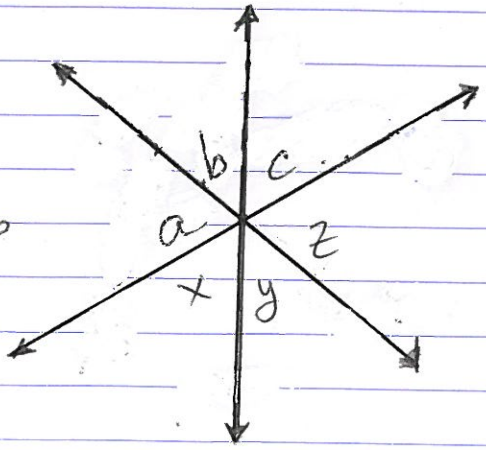
This angle can be called
$\angle BAC$ or $\angle CAB$

The common endpoint or
Vertex is always named
in the middle.

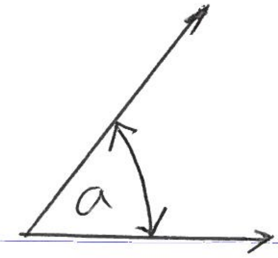


Vertical angles are pairs
of angles when lines
are crossed.

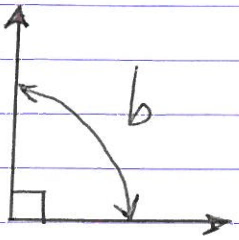
$\angle a = \angle c$
$\angle b = \angle d$



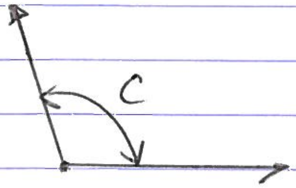
$\angle a + \angle b + \angle c + \angle x$
$+ \angle y + \angle z = 360°$



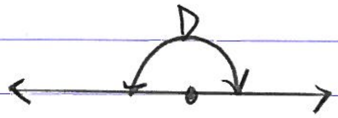
∠a is called an **ACUTE** angle because it is less than 90°



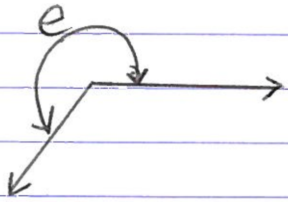
∠b is called a **Right** angle because it equals 90°



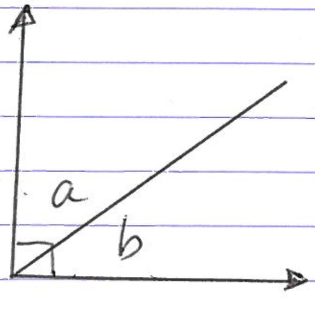
∠C is called an **Obtuse** angle because it is greater than 90°



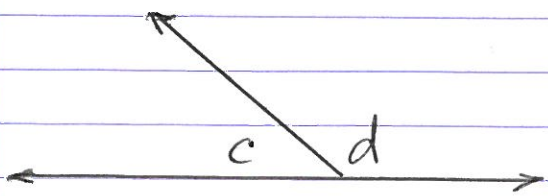
∠d is called a **Straight** angle because it is equal to 180°



∠e is called a **Reflex** angle because it's greater than 180°



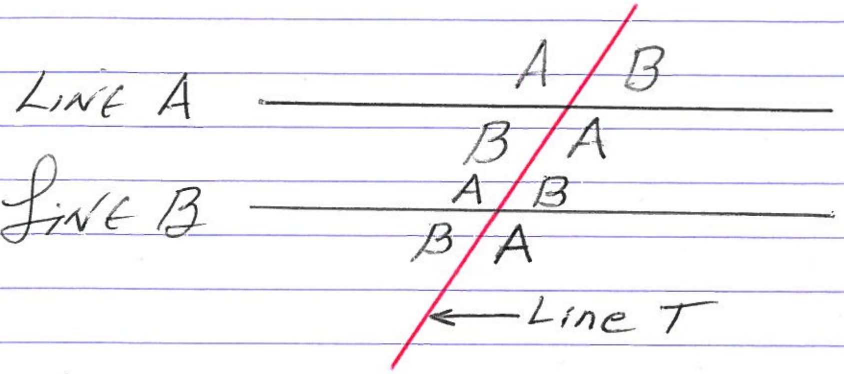
If ∠a + ∠b = 90°, then these angles are called **ComplimenTary** angles



If ∠c + ∠d = 180° then these angles are called **SupplemenTary** angles

# Parallel Lines

Lines are parallel if they are equally spaced apart and will never cross.



Line A ∥ Line B

Line T is called a TRANSVERSAL

When two parallel lines are crossed by a transversal, then certain angle relationships are created.

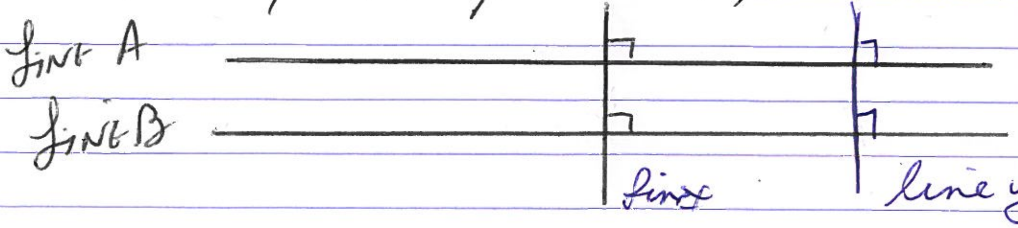
All the angles labelled A are equal

All the angles labeled B are equal
The larger looking angles are equal & smaller ∠'s equal.

Any A plus B angles total 180°

## Perpendicular lines and Parallel lines

If two lines are ⊥ to the same line then they are parallel to each other



If line A is parallel to line B, then all perpendicular lines to those two parallel lines are parallel

# Polygons

Polygons are closed figures created by the intersection of 3 or more lines

TRIANgles are polygons with 3 sides

Quadralerals have 4 sides

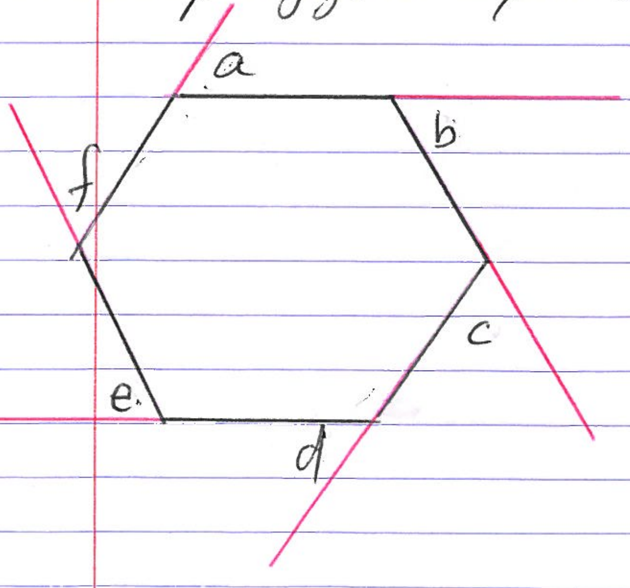
Pentagons have 5 sides

Hexagons have 6 sides

anything beyond 6 sides, polygons are called by how many sides they have

A REGUlAR Polygon has equal measures of its sides and angles

The sum of all the exterior angles of a polygon equals 360°



$a+b+c+d+e+f = 360°$

FOR REGUlAR Polygons ONly → EACH EXTERIOR angle $= \frac{360}{N}$

N = Number of Sides

# Polygons

The sum of ALL interior angles of any polygon equals

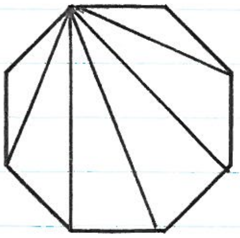
$(n-2)\,180°$   n = number of sides

The exact measure of one angle in a regular polygon equals

$$\frac{(n-2)\,180°}{n}$$ n = number of sides

: Note: You can use this formula to find the average measure of one angle in any polygon

If you forget these formulas, you can draw diagonals from one vertex and count up how many triangles are formed and multiply by 180°



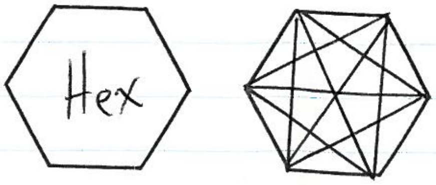
8 sided figure with 6 Δ's formed

$6 * 180° = 1{,}080°$ total

Formula for single ∠ $\frac{(n-2)\,180}{n} = \frac{(8-2)\,180}{8} = 135°$

The total number of diagonals in a polygon equals

$$\frac{n(n-3)}{2}$$

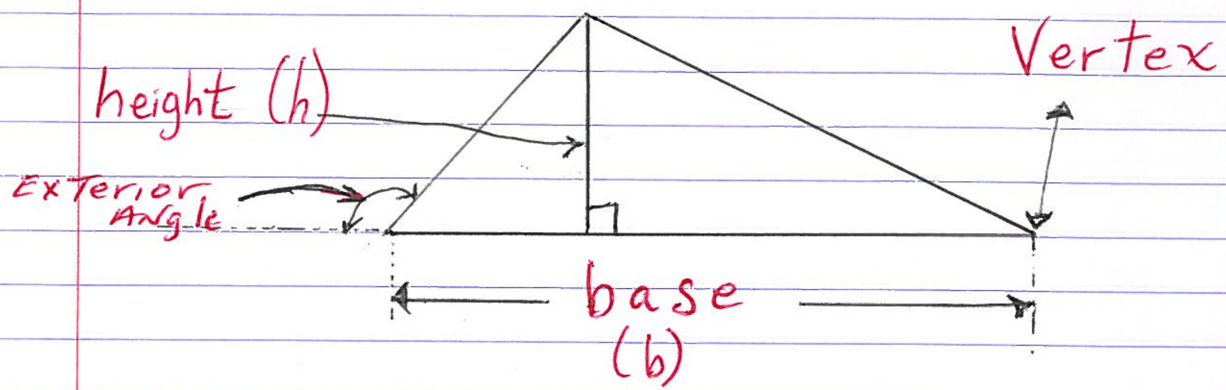


$\frac{6(6-3)}{2} = 9$ diagonals

# Triangle

A triangle is a closed figure with three sides.

Given three line segments of equal or varying lengths, one and only one triangle can be formed.



## TRIANGLE INEQUALITY THEORM

$a < b + c$ – the sum of any two sides is always greater than the length of the third side

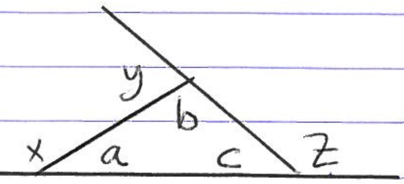
– In other words, each side is shorter than the sum of the lengths of the other two sides.

The sum measures of a triangle total 180°

$y + b = 180$

$x + a = 180$

$z + c = 180$



$X + Y + Z = 360°$

$a + b + c = 180$

$(y + b) + (x + a) + (z + c) = (180)3$

$(a + b + c) + (y + x + z) = 540$

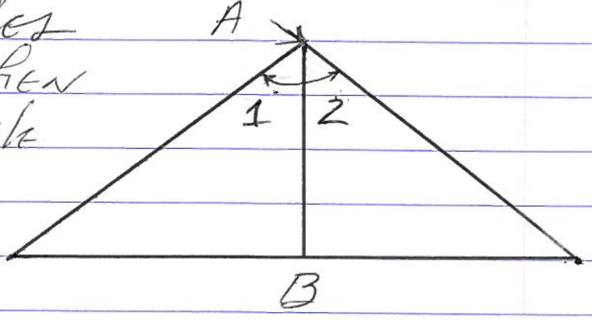
$$\begin{array}{r} 180 + y + x + z = 540 \\ -180 \qquad\qquad\quad -180 \\ \hline y + x + z = 360 \end{array}$$

# General information and definitions about Triangles

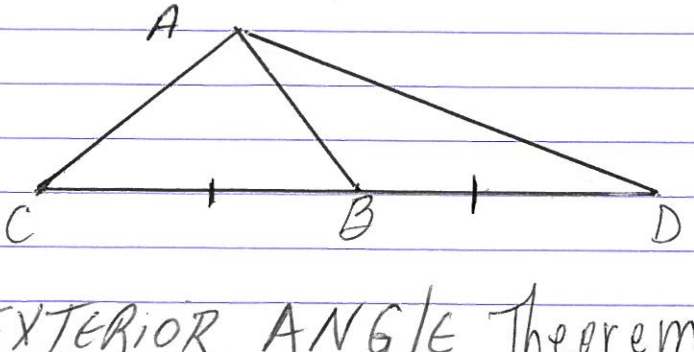
If the values of two angles are known, the 3rd angle can be found by substracting the sum of the known angles from 180°

The largest side of a Δ is opposite the largest angle, the medium side is opposite the middle sized angle, the smallest side is opposite the smallest angle

Angle Bisector
- A line that bisects or breaks into two equal angles
- If ∠1 = ∠2, then line AB is an angle bisector and vice versa.

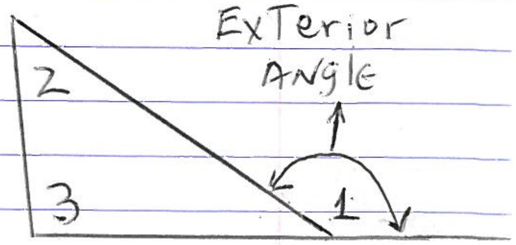


MEDIAN
- Line drawn from any angle to the midpoint of the opposite side.



If $\overline{CB} = \overline{BD}$, then line $\overline{AB}$ is a median.

EXTERIOR ANGLE Theorem
- Formed by extending one side of triangle
- E.A. is equal to the sum of remote interior angles



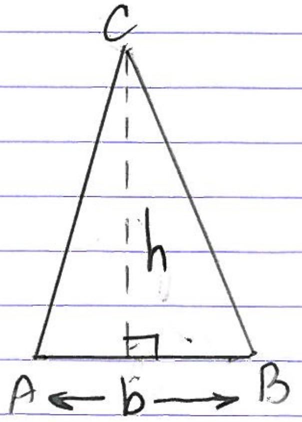
$\angle 1 = \angle 2 + \angle 3$

# Types of Triangles

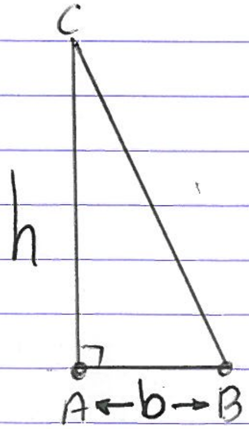
| Angles | Sides |
|---|---|
| Acute Δ's have all 3 angles less than 90° | Scalene Δ's have No sides equal |
| Right Δ's have at least one 90° | Isosceles Δ's have at least two sides equal. |
| Obtuse Δ's have one angle greater than 90° | Equilateral Δ's have all sides equal. - Also called equiangular because all angles are equal. |

No matter what type of Δ it is, All triangles have the following AREA FORMULA

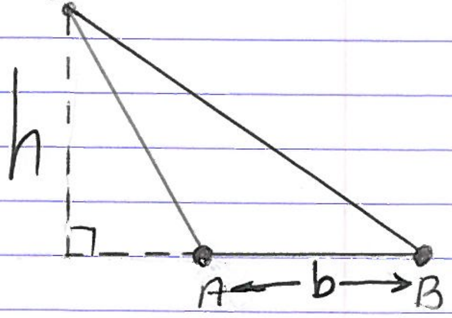
$$\text{Area} = A = \frac{\text{Base} \cdot \text{Height}}{2}$$



ACUTE



RIGHT

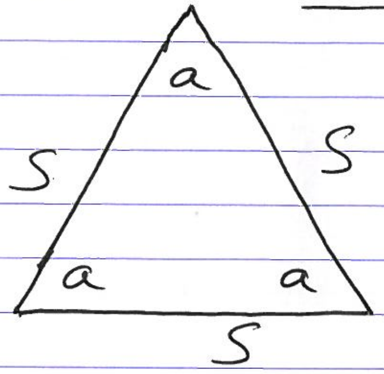


OBTUSE

HEIGHT is a ⊥ line drawn up from base to its opposite vertex

# Formulas and info for the Different Types of Triangles

## EQUILATERAL Δ's



All Sides (s) Equal

All angles (a) Equal

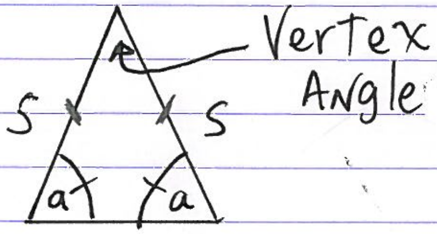
All angles equal 60°

$\text{Perimeter} = 3S$

$$\text{AREA} = \frac{S^2\sqrt{3}}{4}$$

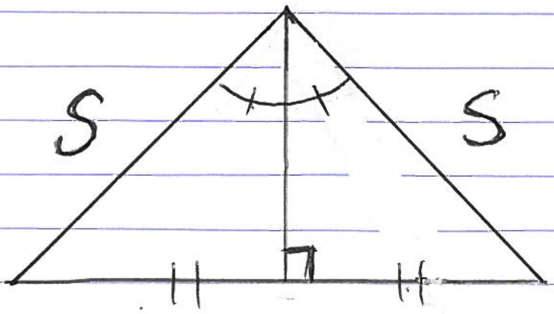
## ISOSCELES Δ's

ISO Δ's have base Angles equal. Therefore, the opposite sides are equal.



ONE Base angle equals $\frac{180 - \text{Vertex}}{2}$

If $\overline{AB} \cong \overline{AC}$, then corresponding base angles are equal.

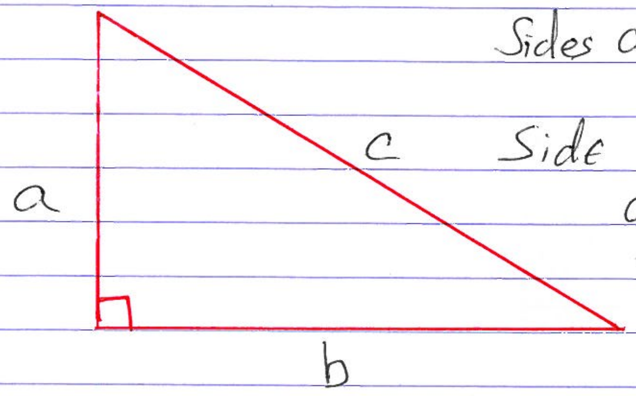


ALTITUDE BISECTS BASE AND VERTEX ANGLE

Use this to easily find area of iso Δ.

# Right Triangles

there's a wealth of knowledge on right triangles. The most important info for the ACT is listed here

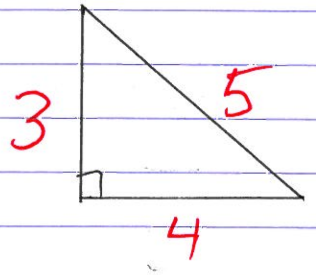


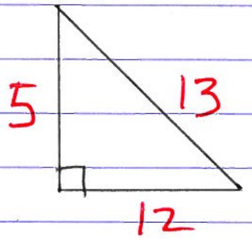
Sides a + b are called **legs**

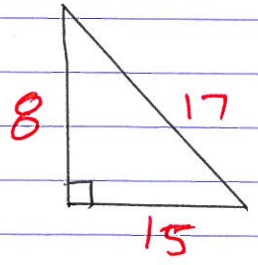
Side C is opposite the right angle as is called the **hypotenuse**. It is always the longest side of the triangle

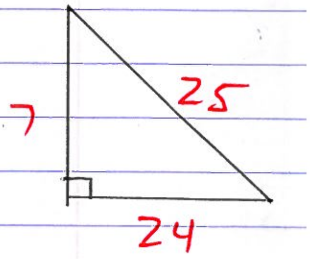
**Pythagorean Formula** $a^2+b^2=c^2$

Know Pythagorean Triples → very useful





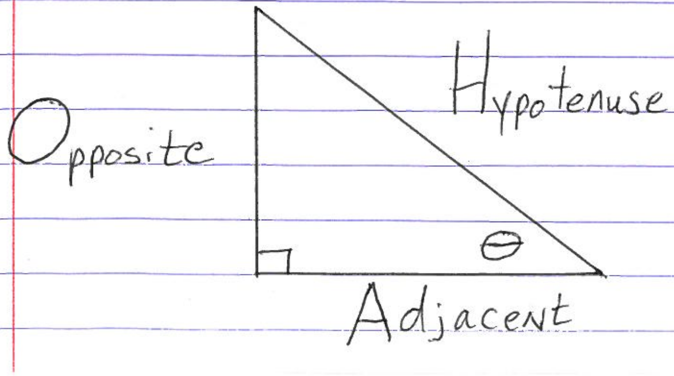




9, 40 + 41 is another one

All multiples of these such as 6, 8, 10 or 16, 30, 34 work as well

**SOHCAHTOA** - ratio of sides in relation to non-right angle angle



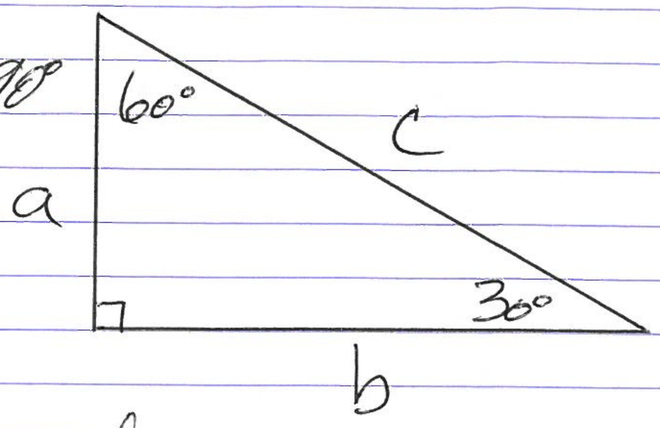
$$\text{Sine } \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H}$$

$$\text{Cosine } \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{A}{H}$$

$$\text{Tangent } \theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{O}{A}$$

# Special Right Triangle Relationships

The 30°, 60°, 90° Right Triangle



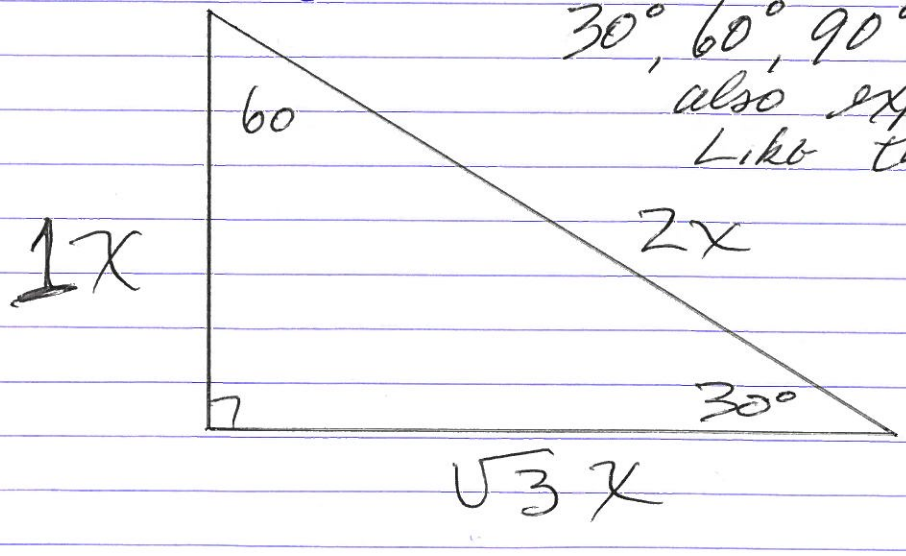
In Terms of the Shortest side a,

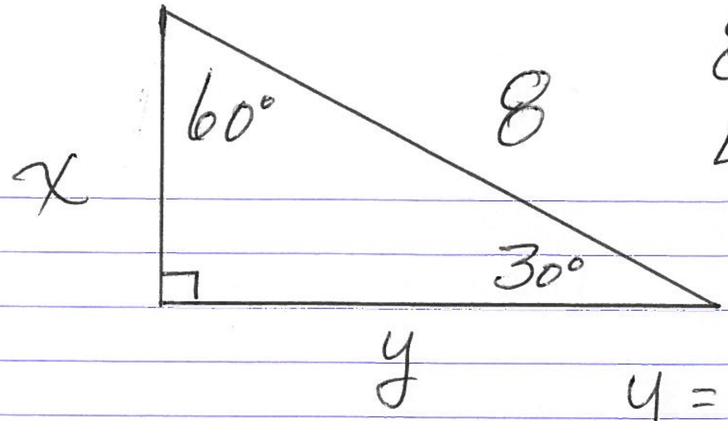
$$a = a = \frac{1}{2}c$$
$$b = a\sqrt{3} = \sqrt{3}c/2$$
$$c = 2a$$

Each side equals this in terms of each other

$$a = \frac{c}{2} = \frac{b}{\sqrt{3}} \qquad b = \frac{c\sqrt{3}}{2} = a\sqrt{3} \qquad c = 2a = 2\frac{b}{\sqrt{3}}$$

30°, 60°, 90° are also expressed Like this



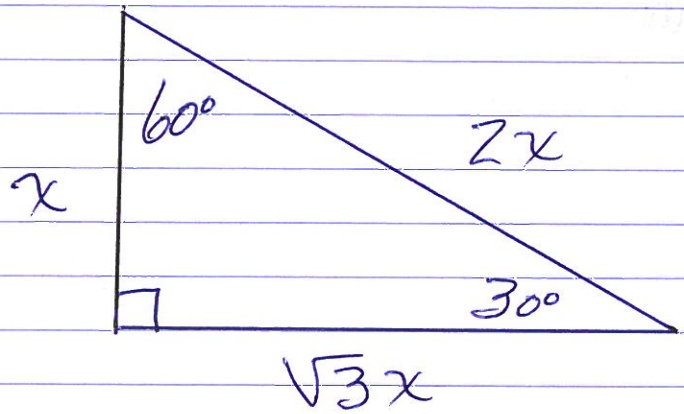


$8 = 2x$

$4 = x$

$$y = \frac{8\sqrt{3}}{2} = 4\sqrt{3}$$

or go Directly $y = x\sqrt{3}$ since $x = 4$ then $y = 4\sqrt{3}$



$$\sin 30° = \frac{x}{2x} = \frac{x}{x} \cdot \frac{1}{2} = \frac{1}{2}$$

$$\cos 30° = \frac{\sqrt{3}\,x}{2x} = \frac{\sqrt{3}}{2}$$

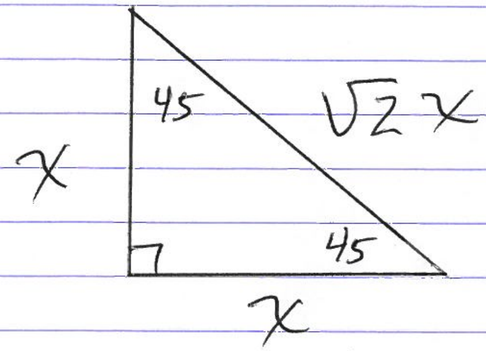
$$\tan 30° = \frac{X}{\sqrt{3}x} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sin 60° = \frac{\sqrt{3}\,x}{2\,x} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{x}{2x} = \frac{1}{2}$$

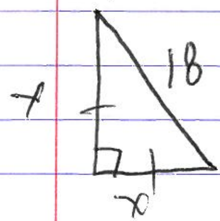
$$\tan 60° = \frac{\sqrt{3}x}{x} = \sqrt{3}$$

# 45°, 45°, 90° Relationships

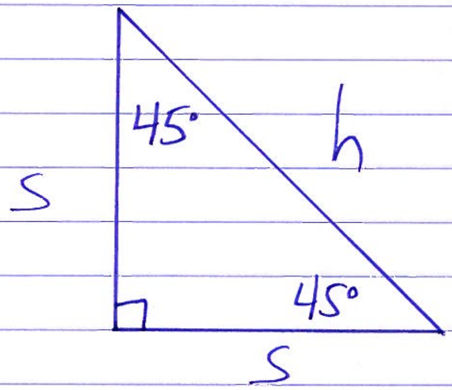


If $x = 8$, then the hypotenuse is $\sqrt{2} \cdot 8$ or $8\sqrt{2}$

Ex.



$x$ = half $18 * \sqrt{2}$



$h = S\sqrt{2}$

$S = \frac{h\sqrt{2}}{2}$

If $h = 4$, then $S = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

If $S = 4$, then $h = 4\sqrt{2}$

$\sin 45° = \frac{S}{\sqrt{2}} = \frac{S\sqrt{2}}{2}$

If $S = 1$, then $\sin 45° = \frac{\sqrt{2}}{2}$

$\cos 45° = \frac{S}{\sqrt{2}} = \frac{S\sqrt{2}}{2}$

If $S = 1$, then $\cos 45° = \frac{\sqrt{2}}{2}$

$\tan 45° = \frac{S}{S} = 1$ No matter what S equals

# Triangle Similarity

Similar Δ's have the same shape, not always the same size.

Two triangles are similar if:

1) Two corresponding angles are =. this is also known as the
No Choice Theorem
2) At least two corresponding sides are in
* the same ratio and the angle forming these two sides is equal.
3) 3 pairs of corresponding sides are in the same ratio.

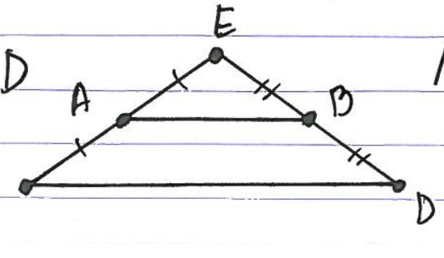
## Summary to prove Triangle Similarity

1) AAA or Angle Angle Angle No choice
2) SAS or Side Angle Side in proportion
3) SSS or Side Side Side in proportion

## Properties of Similar Triangles

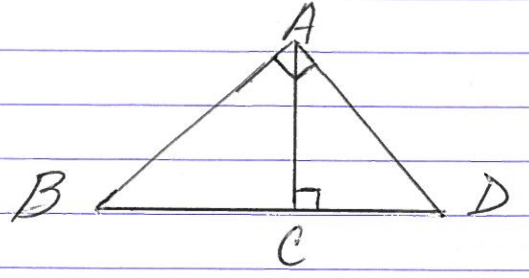
1) Ratio of corresponding sides, medians, altitudes, angle bisectors, heights, and bases are equal in proportion.
Perimeters are in same ratio as sides
2) A line drawn parallel to base creates similar triangles. A line that is drawn from the mid points of two sides is parallel to the 3rd side

ΔAEB ~ ΔCED



Points A and B are midpoints
∴ Line AB is ∥ to CD

3) When an altitude is drawn up from the hypotenuse of a right triangle, similar triangles are formed



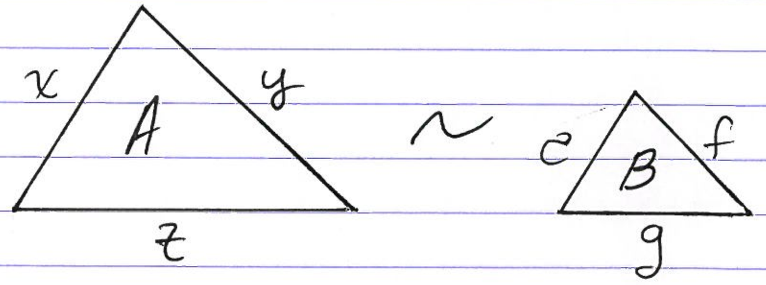
$\triangle ABC \sim \triangle ACD$

$\triangle ABC \sim \triangle ABD$

$\triangle ACD \sim \triangle ABD$

Also → $\frac{AC}{CD} = \frac{CB}{AC}$ or $(AC)^2 = CD \cdot CB$

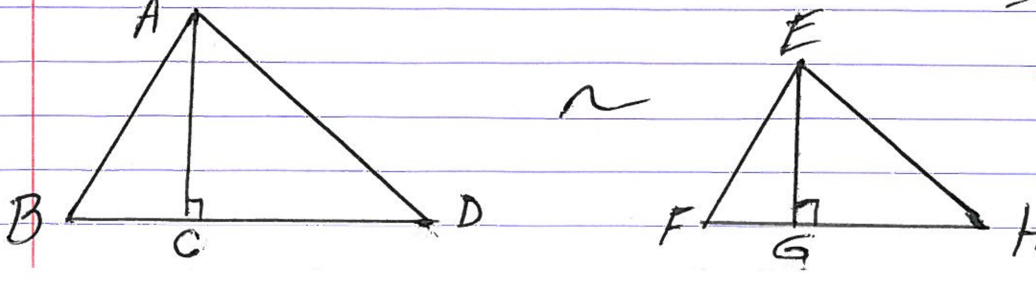
4) When two Triangles are similar they are equal in PROPORTION

If 

, then

$\frac{x}{e} = \frac{y}{f} = \frac{z}{g}$. The proportion is called the Ratio of similarity.

5) Ratio of areas of similar triangles is equal to the square of the ratio of similarity



See Next page for all the relations

#6

5) Continued Altitude and bases are equal in proportion

$$\frac{AC}{BD} = \frac{EG}{FH}$$

---

$$\left(\frac{\text{AREA OF } \Delta ABD}{\text{AREA OF } \Delta FEH}\right)^2 = \left(\frac{AC}{EG}\right)^2 = \left(\frac{BD}{FH}\right)^2$$

OR

Ratio of Areas of Δ's = Square of ratio of Similarity

The following figures are always Similar

1) Equilateral / equiangular triangles
2) 45/45/90 and 30/60/90 Δ's
3) All Squares and all circles
4) Any two regular polygons that have the Same Number of Sides.

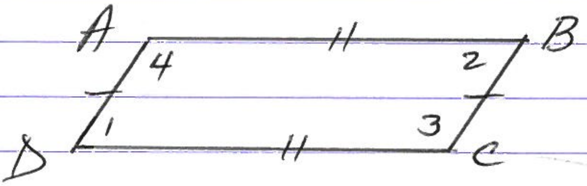
# Quadrilaterals

Quadrilaterals are closed, Two-dimensional figures with four sides created by line segments.

Since any quad. can be split into two triangles and triangles have 180° each, quad's have internal angles totalling 360°.

Most of the questions on quads, deal with parallelograms. Parallelograms have both pairs of opposite sides equal in length and are parallel. Opposite angles are congruent.

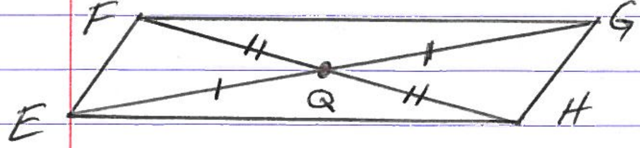
There are different Types of parallelograms that will be discussed later, but for now these are characteristics of all parallelograms.



$\overline{AB} = \overline{DC}$ $\angle 1 = \angle 2$

$\overline{AD} = \overline{BC}$ $\angle 4 = \angle 3$

$\angle 1 + \angle 2 = 180°$ $\angle 3 + \angle 4 = 180°$ All 4 ∠'s = 360°

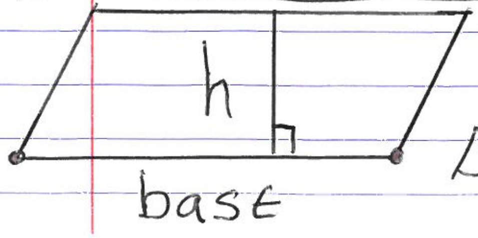


Diagonals bisect each other

$\overline{EQ} = \overline{QG}$ and $\overline{FQ} = \overline{QH}$

Congruent triangles are formed $\Delta FQE = \Delta GQH$

$\Delta FQG = \Delta EQH$

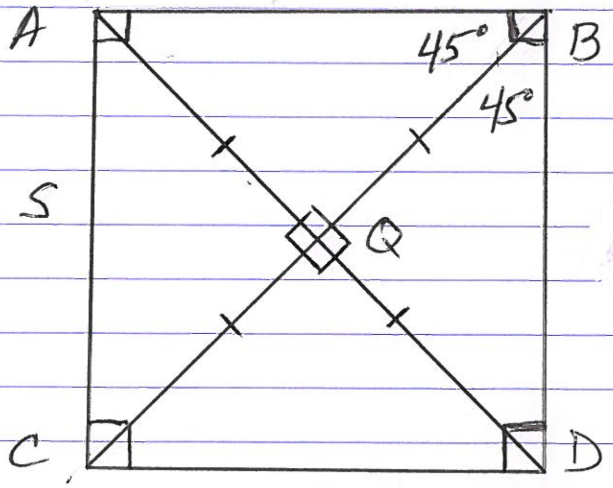


AREA = Base • Height

# Squares are special parallelogram

All squares have:

- 4 equal sides and 4 right angles
- diagonals which are perpendicular bisectors of each other.
- diagonals create 4 right angles
- perimeter equals 4s (s = side)
- area equals $s^2$ or $\frac{d^2}{2}$ (d = diagonal
- diagonals create 4 congruent triangles.
- diagonal length = $s\sqrt{2}$
- Has the characteristics of rectangle or rhombus because a square is both of these.



**Summary**

$$\text{Area} = s^2 = (\text{Any side})^2 = \frac{d^2}{2}$$

$$\text{Perimeter} = 4s$$

$$\text{Diagonal length} = s\sqrt{2}$$

- Rhombus is an equilateral Parallelogram

# Rhombus is a square that's been thru a hurricane. Rhombi have:

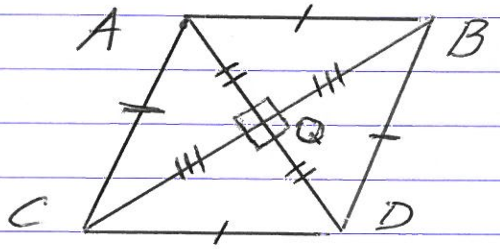
- Four equal sides
- Two sets of supplementary angles not necessarily equal to 90°
- diagonals that are perpendicular bisectors of each other just squares.

# Rhombus (cont.)

- Area formula is the same as any parallelogram → base • height or

$$\frac{d^1 \cdot d^2}{2} = \text{Area} = \frac{\text{Product of diagonals}}{2}$$

- Perimeter = 4s just like square
- Diagonals divide rhombus into 4 congruent right triangles
- All squares are rhombi, some rhombi are squares.



$\overline{CB} \perp \overline{AQ}$

$\overline{AD} \perp \overline{QB}$

## Summary

4 sides equal

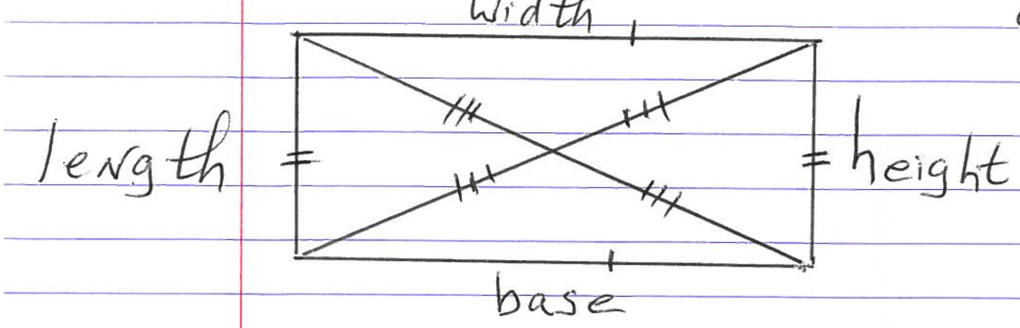
$\overline{AQ} = \overline{QD}$ & $\overline{CQ} = \overline{QB}$

$\triangle AQB \cong \triangle BQD \cong \triangle CQD \cong \triangle ACQ$

$$\text{Area} = \text{base} \cdot \text{height} = \frac{d^1 d^2}{2} = \frac{\overline{CB} \cdot \overline{AD}}{2}$$

Perimeter = 4 S

Rectangles are parallelograms with four 90° angles, but unequal sides



All rectangles including squares have the following characteristics:

- opposite sides equal in length
- All angles are 90°
- Sum of angles equal 360°
- perimeter = $2L + 2W$.
- Area = $L \cdot W$ = base · height
- diagonals are equal in length
- diagonals bisect each other
    - perpendicular only if square
- each diagonal creates two congruent triangles each 1/2 the area of the rectangle
- opposing triangles created by diagonals are congruent and isosceles

All squares, rectangles and rhombi are parallelograms.

All squares are rectangles and rhombi

Trapezoids, our next topic, are not parallelograms.

# Trapezoid

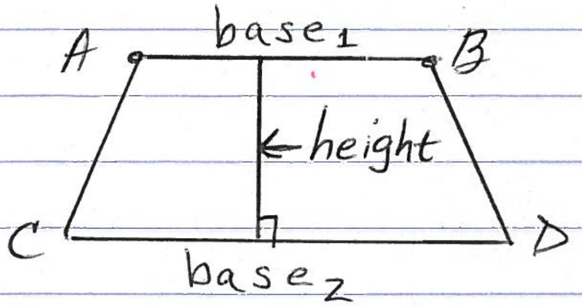
A trapezoid is a quadralateral that has only ONE SET OF PARAllel lines

The bases (Top and bottom) are parallel and are NEVER the same length.

When the sides are equal, it's CALLED an isosceles trapezoid

Diagonals are equal only in isosceles trapezoids. If sides ≠, the diagonals ≠ If diagonals equal, then sides are equal

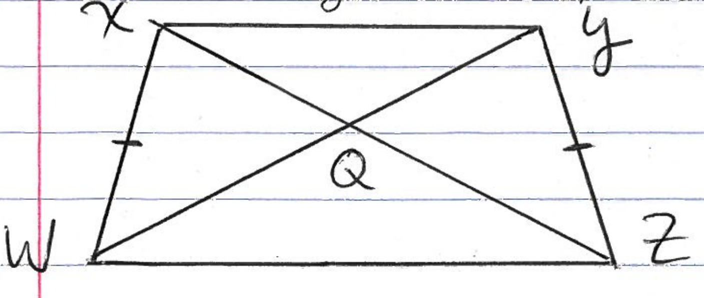
Area equals average of bases times height



height is ⊥ line connecting $\text{base}_1$ and $\text{base}_2$

$$\text{Area} = \frac{AB+CD}{2} \cdot h$$

IN isosceles trapezoids, special relationships exist between diagonals and bases.

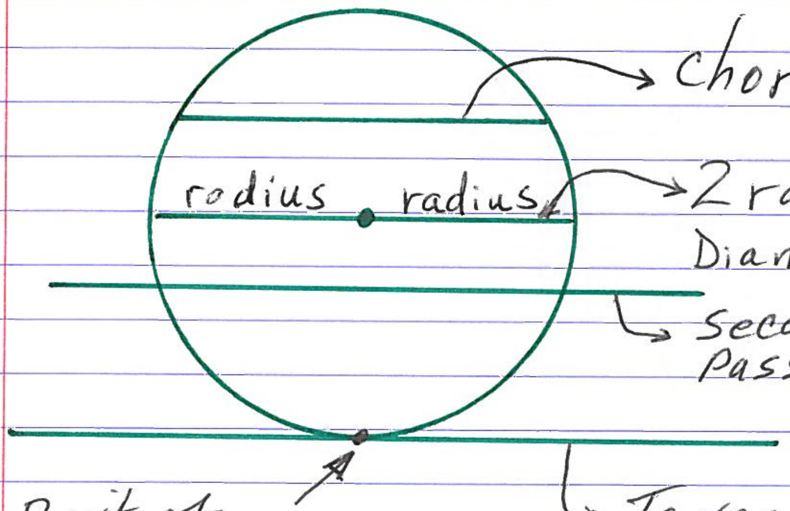


$$\frac{XQ}{QZ} = \frac{XY}{WZ} \text{ and } \frac{WQ}{QY} = \frac{WZ}{XY}$$

If $XZ = WY$, then $XW = YZ$

# Circles

A circle is defined as a set of points equidistant from a fixed center point. The distance from the **CENTER** to any point on the circle is called the **radius**.



chord is line with endpoints on circle

2 radii = 1 diameter
Diameter is longest chord

secant is a chord that passes thru circle

Point of tangency

Tangent is a line that shares one point with the circle

**Radius** is line segment from center to any point on the outside of the circle

**Diameter** is a special chord that passes thru the center.

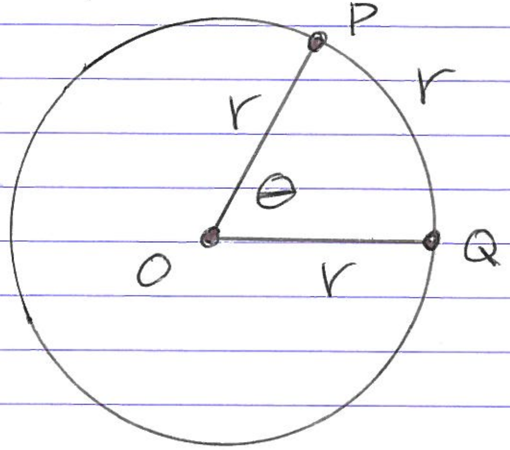
$$D = 2r \qquad r = \frac{D}{2}$$

**Circumference** is simply the perimeter or distance around.

$$\pi = \frac{\text{Circumference}}{\text{Diameter}} \therefore C = \pi D = 2\pi r$$

# Area of Circle = $\pi r^2$

A **radian** is the amount of angle measurement when the arc length equals the radius

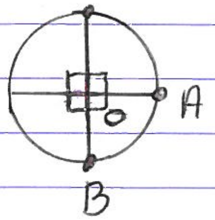


If $\overline{OP} = \widehat{PQ} = \overline{OQ}$, then

$\theta = 1$ radian.

$$\frac{2\pi r \text{ radians}}{360°} = \frac{\text{radian measure}}{\text{degree measure}}$$

By convention circles have a total of 360°. Therefore, four central angles of 90° each can be made around the center



These four angles are called central angles. $\widehat{AB}$ is called an arc. Angle O creates 90° of arc.

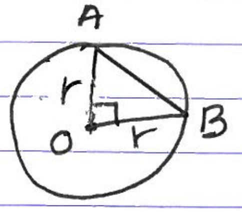
If $\angle O$ is central angle, then

$$\angle O \text{ degrees} = \widehat{AB} \text{ degrees}$$

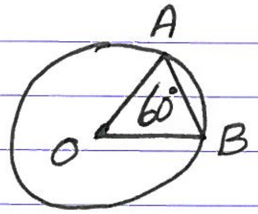
$$\frac{\text{Central Angle}}{360°} = \frac{\text{arc length}}{\text{circumference}}$$

Circles have 360° degrees or $2\pi$ radians depending on which measuring method is used.

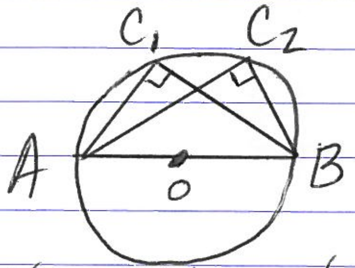
# Triangles Within Circles



Right central angles create 45°, 45°, 90° triangles. Therefore, Area of $\Delta AOB = \frac{r^2}{2}$ and $\overline{AB} = r\sqrt{2}$



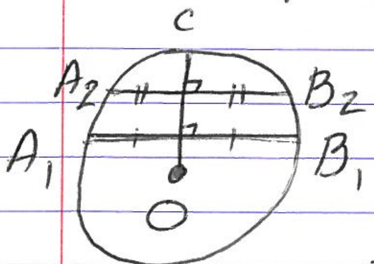
If central angle equals 60°, then an equilateral triangle is formed. $\overline{AB}$ = radius



If one side of inscribed triangle is the diameter, then a right triangle is always formed.

No matter where C is, a right angle is always created.

More important information regarding Circles

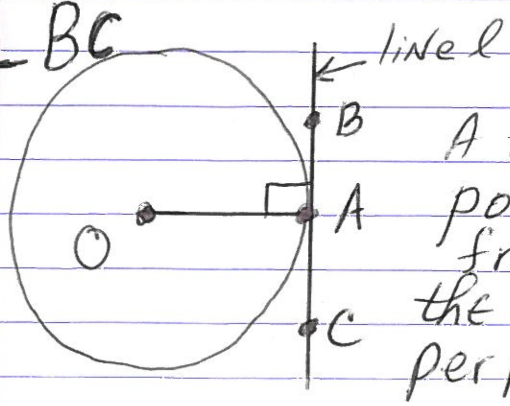


When a radius is perpendicular to a chord, this radius bisects the chord

If $\overline{OC}$ is ⊥ AB, then AB is bisected.

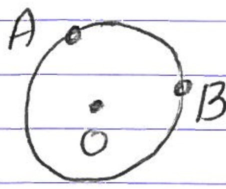
If $\overline{AB}$ is bisected by radius $\overline{OC}$, then $\overline{OC}$ is ⊥ to AB

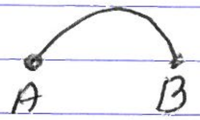
$\overline{OA} \perp \overline{BC}$



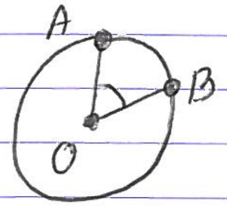
Line l is a tangent line. A tangent line shares ONE point with a cirle. A line from this shared point to the circle's center is always perpendicular to the tangent line

# More about arcs in circles

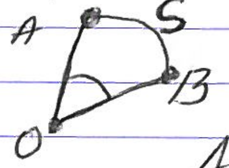




$\overset{\frown}{AB}$ is an **arc** of circle O.



If $\angle AOB$ is a central angle, then



is a **Sector** of circle O.

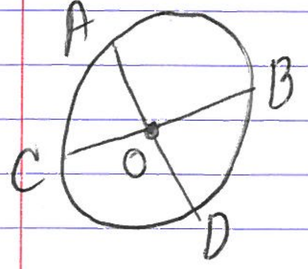
$$\text{Area of Sector } AOB = \frac{\angle AOB^\circ}{360^\circ} \cdot \pi r^2$$

$$S = \text{Length of arc } AB = \frac{\angle AOB^\circ}{360^\circ} \cdot 2\pi r$$

$S = \theta R$ in radians $\theta =$ central angle

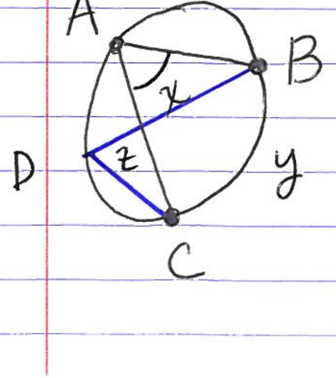
$$\text{Perimeter of entire Sector} = S + 2r$$

$$\text{Area of section of circle with radius } r = \frac{\text{central Angle}^\circ}{360^\circ} \cdot \pi r^2$$



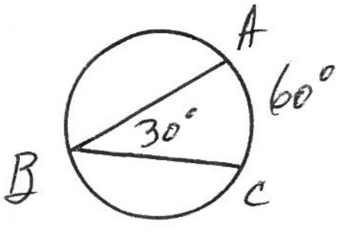
$AO \cdot OD = CO \cdot OB$ or

$\frac{CO}{AO} = \frac{OD}{OB}$ O does not have to be center



If $\overline{AB} = \overline{AC}$, then $\overset{\frown}{AB} = \overset{\frown}{AC}$

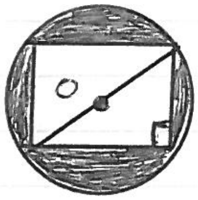
$\angle x^\circ = \frac{1}{2} y^\circ$. Since $\angle x + \angle z$ create the same intercepted are, $\angle x = \angle z$



If ∠ABC = 30°, then arc AC is doubled or 60°

If $\overset{\frown}{AC}$ is 60° then inscribed angle is halved or 30°

INSCRIBED <u>Square</u> in a circle

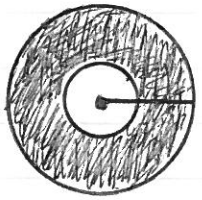


The difference between the circle's area and the square's area equals the shaded region

$$\underset{\text{Area of cirle}}{\pi r^2} - \underset{\text{area of square}}{2r^2} = \text{Shaded region}$$

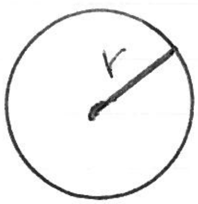
Area of one shaded section $\frac{1}{4}(\pi r^2 - 2r^2)$

Ratio of area of square to circumference equals $\frac{2}{\pi}$



Area of Shaded region equals area of large circle minus area of smaller circle

Ratio between a circle's radius to circle's area or circle's circumference to its area is <u>EXPONENTIAL</u> not linear

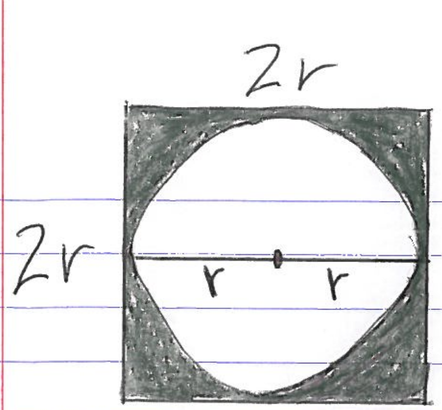


$r = 3$ then area $= \pi(3)^2$
$= 9\pi$

$\frac{\text{radius}}{\text{area}} = \frac{3}{9\pi}$ $9 = 3^2$

$\frac{\text{circumference}}{\text{area}}$

$\frac{2\pi r}{\pi r^2} = \frac{6\pi}{9\pi} = \frac{2}{3}$



Circumscribed square around circle

Each side of square = 2r

Area of Square = $4r^2$

Area of Circle = $\pi r^2$

Ratio of Area of Square to area of Circle

$$\frac{4r^2}{\pi r^2} = \frac{4}{\pi}$$

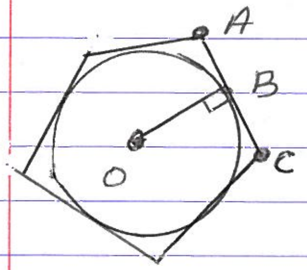
Difference of areas (shaded region) =

$$4r^2 - \pi r^2 = r^2(4-\pi)$$

Area of one shaded section = $\frac{1}{4} r^2(4-\pi)$

---

When regular polygons are circumscribed around circles:

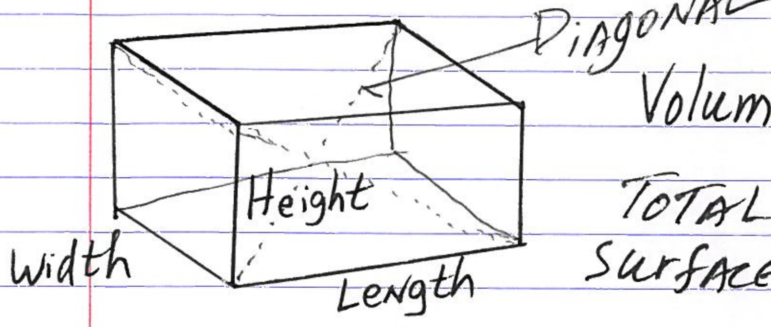


Points of tangency are perpendicular to center and bisect the sides and opposite angles.

If $\overline{OB} \perp \overline{AC}$, then $\overline{AB} = \overline{BC}$

If AC is tangent line to circle O, then radius $\overline{OB}$ is $\perp$ to $\overline{AC}$

# Three Dimensional Geometric Figures



Volume = $L \cdot W \cdot H$

Total Surface Area $= 2(LW + LH + WH) = 2LW + 2LH + 2WH$

Diagonal = D    $d^2 = L^2 + w^2 + h^2$

D = longest line segment in rectanglar solid.    $d = \sqrt{L^2 + w^2 + h^2}$

Cube is a rectangUlar solid with all sides =

Volume of cube $= S^3$

$S = \sqrt[3]{\text{Volume}}$

Surface Area of cube $= 6s^2 = \left(\sqrt[3]{\text{Volume}}\right)^2$

Cube has 6 faces 8 Vertices 12 edges

Diagonal of Cube = Edge length $\cdot \sqrt{3}$

because $d^2 = 3 \cdot (\text{edge})^2$

Volume of Prism = Area of base * height

Volume of sphere $= \frac{4}{3}\pi r^3$

Surface area of sphere $= 4\pi r^2$

Volume of Cylinder $= \pi r^2 h$    h=height

Volume of right circular cone $= \frac{h(\pi r^2)}{3}$

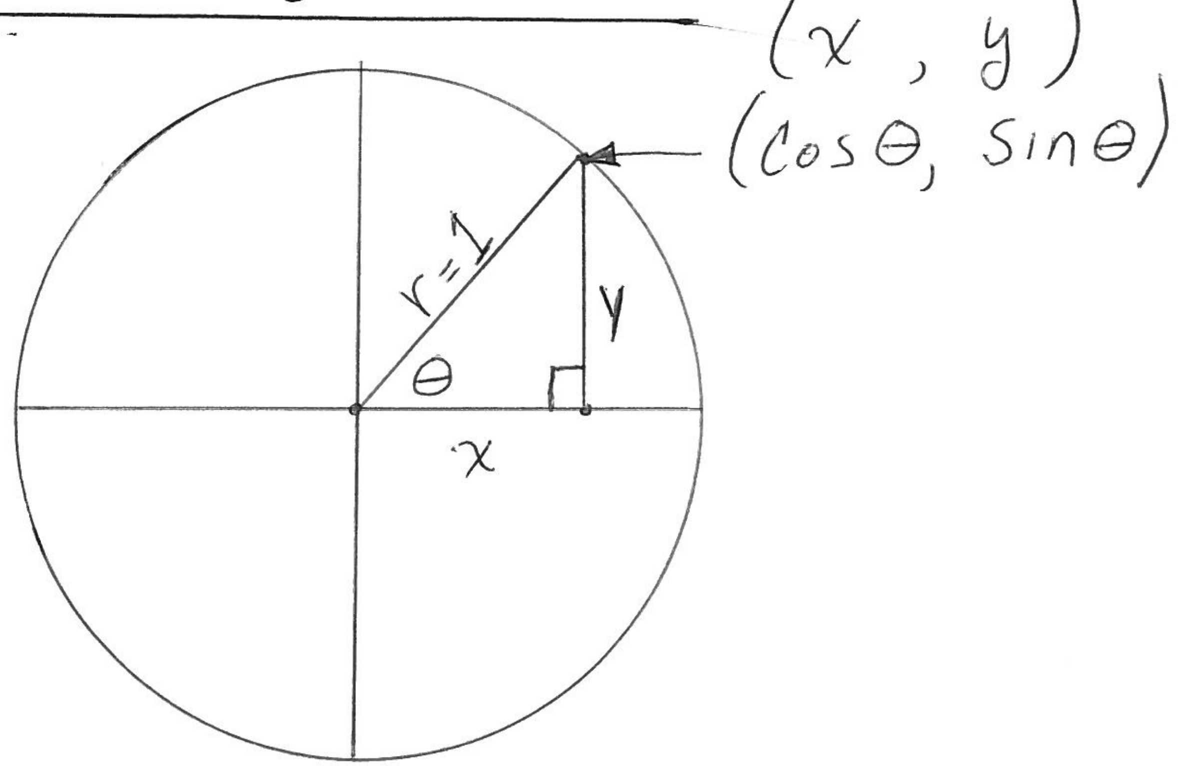
Surface area of right circular cone

$SA = \pi r^2 + \pi r s$    s = slant height $= \sqrt{r^2 + h^2}$

# Trigonometry

TrigoNometry Means "triangle Measurement." TrigoNemetric functions use ratios of the sides of right triangles.

## The Unit Circle radius = 1



$$\text{Sine } \theta = \frac{y}{1} = y = \frac{\text{opposite side}}{\text{Hypotenuse}} = \sin\theta$$

Sine 0° = 0
Sine 90° = 1

$$\text{Cosine } \theta = \frac{x}{1} = x = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \cos\theta$$

Cosine 0° = 1
Cosine 90° = 0

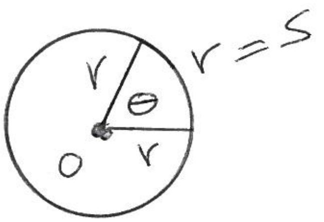
$$\text{Tangent} = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$$

Tan 0° = 0
Tangent 90° → undefined

# Radians

**Radians** are a measure of the distance along the surface of a circle's circumference.

When a central angle creates an arc that is equal to the length of the radius, the angle measure is called one RADIAN



When the circle's arc (s) equals the radius then θ equals one radian.

$$\boxed{\frac{S}{R} = \text{MEASURE of } \theta \text{ in radians}}$$

Since a circle's circumference equals $2\pi r$ there are $2\pi$ radians in a circle.

$360° = 2\pi$ radians

$180° = \pi$ radians

$90° = \frac{\pi}{2}$ radians

$45° = \frac{\pi}{4}$ radians

$1° = \frac{\pi}{180}$ radians

To convert from degrees to radians

$\longrightarrow$ Multiply by $\frac{\pi}{180^\circ}$ $\longleftarrow$

Example: Translate 60° to radians

$$60^\circ * \frac{\pi}{180^\circ} = \frac{60^\circ}{180^\circ}\pi = \frac{\pi}{3} \text{ radians}$$

To convert from radians to degrees

$\longrightarrow$ Multiply by $\frac{180^\circ}{\pi}$ $\longleftarrow$

Example: Translate $\frac{\pi}{6}$ radians to degrees

$$\frac{\pi}{6} * \frac{180^\circ}{\pi} = \frac{180^\circ}{6} * \frac{\pi}{\pi} = 30^\circ$$

## Summary

one radian equals the angle measurement when the radii equals the arc length

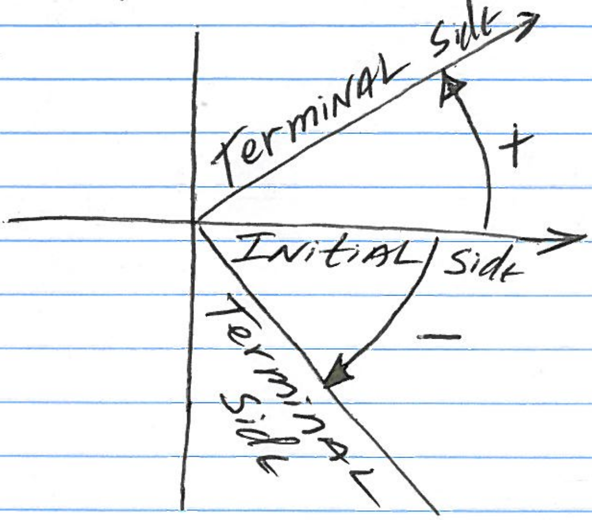
the arc length of a circle divided by radius equals 2π radians = 360°

$$1 \text{ Radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}$$

$$1 \text{ Degree} = \frac{2\pi \text{ RADIANS}}{360^\circ} = \frac{\pi \text{ Radians}}{180^\circ}$$

When thinking of angles and trigonometric ratios, think of the radius as the second hand on a watch.

Positive angles are created when the radius moves counter clockwise up from the x-axis. Angles are negative when they move clockwise when x-axis.



Angles are positive when they go counter-clockwise.

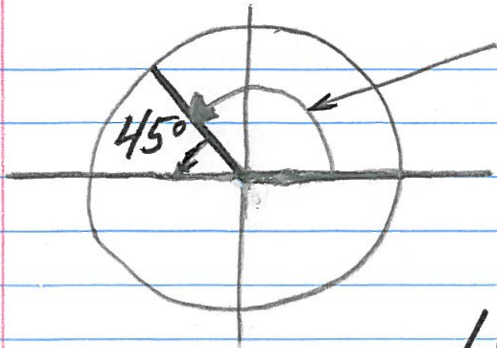
Angles are negative when the radius goes clockwise

Trigonometric ratios of Sin, Cos, etc are all based on angles less than 90° or equal to 90°. When dealing with obtuse angles, we need to use reference angles.

A reference angle is an acute angle closest to the x-axis.

To find the reference angle from an obtuse angle subtract the obtuse angle measurement from 180° or π radians.

Example for finding reference angle from obtuse angle.



original obtuse angle is 135°

The 45° angle is closest to x-axis.

$$180° - 135° = 45°$$

So an obtuse angle of 135° would have the same trigonometric ratios as a 45° angle.

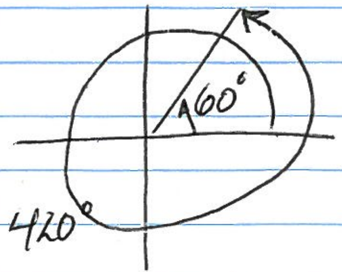
---

Using the second hand on a watch analogy again, the radius could keep spinning around the center indefinitely.

When this happens, angles become greater than 360° or $2\pi$ radians. We call these angles Coterminal Angles.

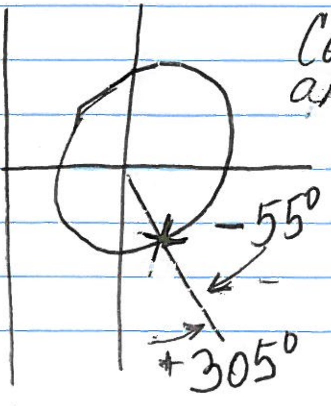
Coterminal Angles are found by adding or subtracting even integer multiples of 360° or $2\pi$ radians.

Coterminal angles have the same initial and terminal side, but could have different rotations.



Angle coterminal with 420° is

$$420° - 360° = 60°$$



Coterminal angle with -55° is 305°

$$-55° + 360 = 305°$$

# ACT TEST Example for Coterminal Angles.

What is the smallest possible positive angle that is Coterminal with $\frac{75\pi}{4}$?

A) 120° B) $\frac{\pi}{4}$ C) $\frac{-\pi}{4}$ D) $\frac{3}{4}\pi$ E) $\frac{11\pi}{4}$

ANSWER: To find coterminals you have to subtract a multiple of $2\pi$ radians. Since the angle you're given is a multiply of $\frac{1}{4}$, you need to subtract a $\frac{1}{4}$ multiple of $2\pi$.

$2\pi$ is the same thing as $\frac{8\pi}{4}$

So what's the smallest integer multiple of $\frac{8\pi}{4}$ that could subtract from $\frac{75\pi}{4}$? $9 \times 8 = 72$

$$\frac{75\pi}{4} - 9\left(\frac{8\pi}{4}\right) = \frac{3\pi}{4}$$ choice D

Note: $\frac{3\pi}{4}$ radians is an obtuse angle because its greater than $\pi/2$ radians or 90°

The reference angle for $\frac{3\pi}{4}$ radians is

$$\pi \text{ radians} - \frac{3\pi}{4} = \frac{\pi}{4} \text{ radians}$$

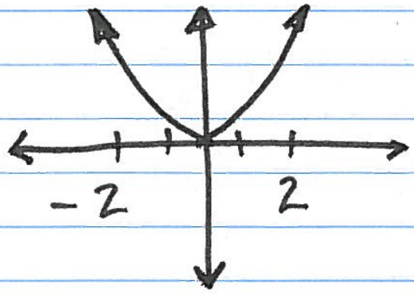
# EVEN and ODD FUNCTIONS

If the input AND the input's inverse produces the same output, then the function is even.

Even when: $f(-x) = f(x)$

Even functions are symetrical to the y-Axis.

A good example of an even function is $f(x) = x^2$



$$f(x) = f(2) = 2^2 = 4 = f(x)$$

$$f(-x) = f(-2) = (-2)^2 = 4 = f(-x)$$

Input and input's inverse produces the same result.

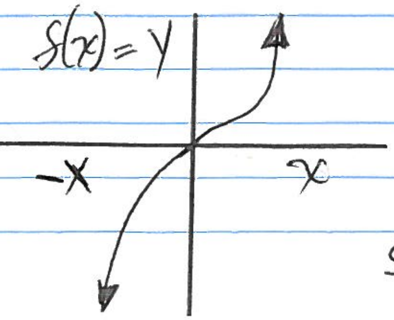
---

If the input's inverse produces a negative of the function, then the function is odd

ODD when: $f(-x) = -f(x)$

A good example of an odd function is

$$f(x) = x^3$$



$$f(-x) = f(-3)^3 = -(3)^3$$

$$= f(-3)^3 = -f(3)^3$$

odd functions are symmetric about the origin

# Trigonometric Identities
## Use Unit Circle for Reference

$$\text{Sine } \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{Y \text{ Value}}{\text{Radius}} = \frac{Y}{R}$$

$$= \frac{1}{\text{Cosecant } \theta} = Y \text{ if radius is ONE.}$$

Positive in Quadrants $\underline{I}$ and $\underline{II}$
Negative in Quadrants $\underline{III}$ and $\underline{IV}$

---

$$\text{Cosine } \theta = \frac{\text{Adjacent}}{\text{hypotenuse}} = \frac{X \text{ Value}}{\text{radius}} = \frac{X}{R}$$

$$= \frac{1}{\text{secant } \theta} = X \text{ if radius is ONE}$$

Positive in Quadrants $\underline{I}$ and $\underline{IV}$
Negative in Quadrants $\underline{II}$ and $\underline{III}$

---

$$\text{Tangent } \theta = \frac{\text{opposite}}{\text{Adjacent}} = \frac{Y \text{ Value}}{X \text{ Value}} = \frac{Y}{x}$$

$$= \frac{\text{Sin } \theta}{\text{Cos } \theta} = \frac{1}{\text{Cot} \theta} = \text{Slope of Radius}$$

Positive in Quadrants $\underline{I}$ and $\underline{III}$
Negative in Quadrants $\underline{II}$ and $\underline{IV}$

$$\text{Cosecant } \theta = \frac{\text{radius}}{\text{y value}} = \frac{r}{y} = \frac{1}{\sin\theta}$$

Positive in I and II
Negative in III and IV

$$\text{Secant } \theta = \frac{\text{radius}}{\text{x value}} = \frac{r}{x} = \frac{1}{\cos\theta}$$

Positive in I and IV
Negative in II and III

$$\text{Cotangent } \theta = \frac{\text{x value}}{\text{y value}} = \frac{x}{y} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$

$$= \frac{1}{\text{slope of radius}}$$

Positive in Quadrants I and III
Negative in Quadrants II and IV

Trigonometry is the math of cycles. It's the analysis of anything that repeats.

When you chart the movement of positions created by the changes in Trig ratios, sine waves are created. On the next page, graphs of the different Trig ratios are charted.

As the radius moves about the center, the sine changes. These changes become the Sine WAV

# The unit circle and graphs of trigonometric functions

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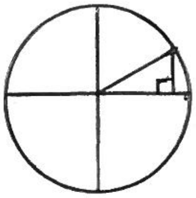
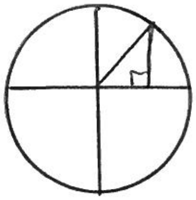
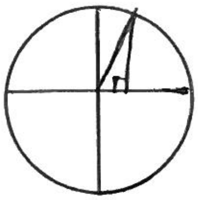
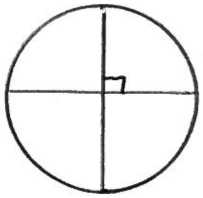
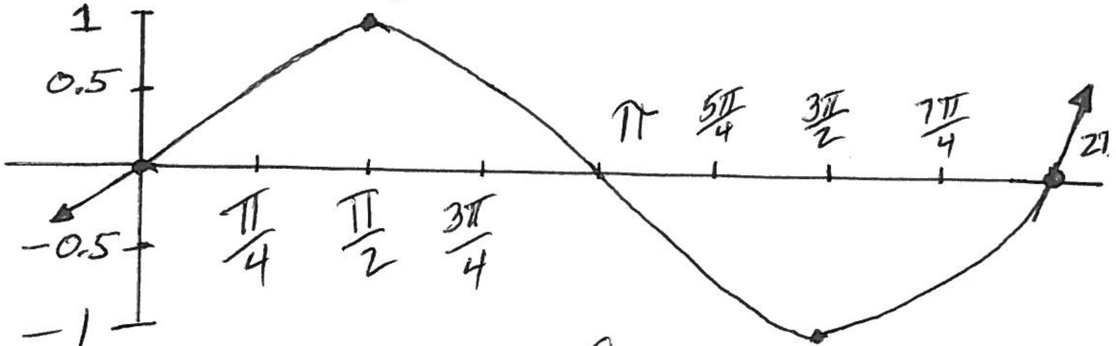
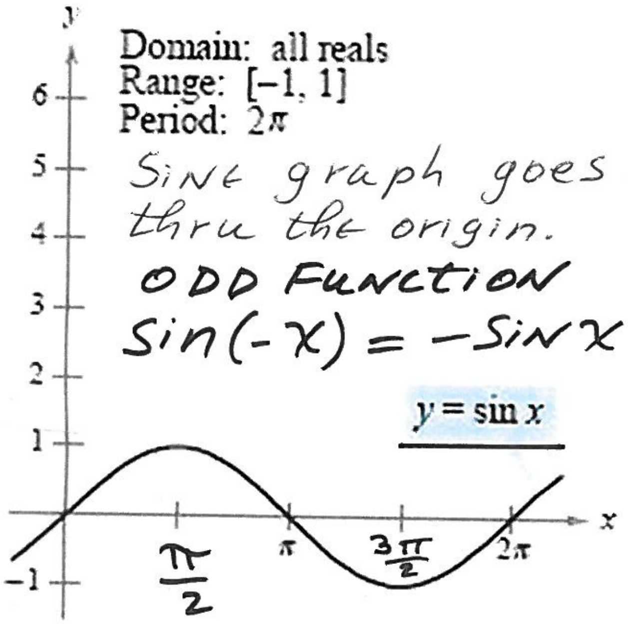
|  |  |  |  |
| $\sin\frac{\pi}{6} = \sin 30^\circ$<br>$= \frac{1}{2}$ | $\sin\frac{\pi}{4} = \sin 45^\circ$<br>$= \frac{\sqrt{2}}{2}$ | $\sin\frac{\pi}{3} = \sin 60^\circ$<br>$= \frac{\sqrt{3}}{2}$ | $\sin\frac{\pi}{2} = \sin 90^\circ$<br>$= 1$ |

| Sine Function | Input | output |
|---|---|---|
| $y = \sin\theta$ | $\frac{\pi}{6}$ or $30^\circ$ | $\frac{1}{2} = 0.500$ |
| $\text{Sine} = \frac{\text{opposite}}{\text{hypotenuse}}$ | $\frac{\pi}{4}$ or $45^\circ$ | $\frac{\sqrt{2}}{2} = 0.707$ |
| | $\frac{\pi}{3}$ or $60^\circ$ | $\frac{\sqrt{3}}{2} = 0.866$ |
| | $\frac{\pi}{2}$ or $90^\circ$ | $1 = 1.000$ |

ON a coordinate graph, the horizontal line represents the inputs and the vertical lines represents the outputs. When you plot the angle measure of the central angle as the inputs (x) and the actual trig ratio produced given that angle, you create a graph that looks like this:



This is the general graph for the sine function
See how the graph is increasing thru the origin.



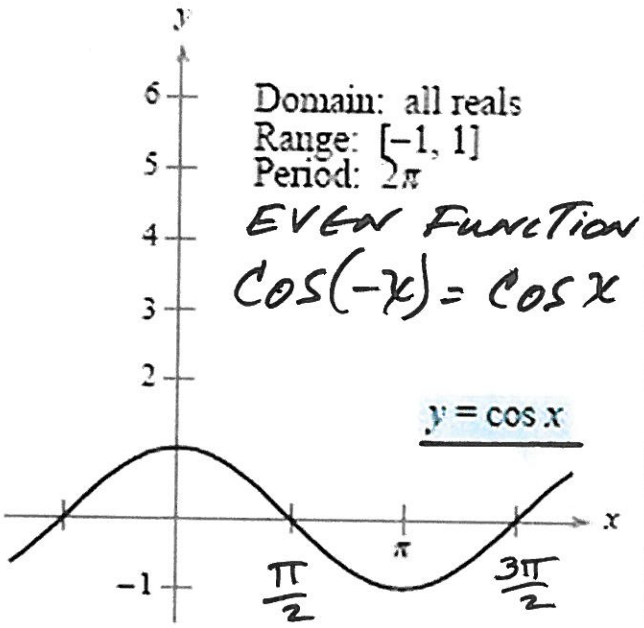
Domain: all reals
Range: [−1, 1]
Period: 2π

Sine graph goes thru the origin.

ODD FUNCTION

$\sin(-x) = -\sin x$

$y = \sin x$



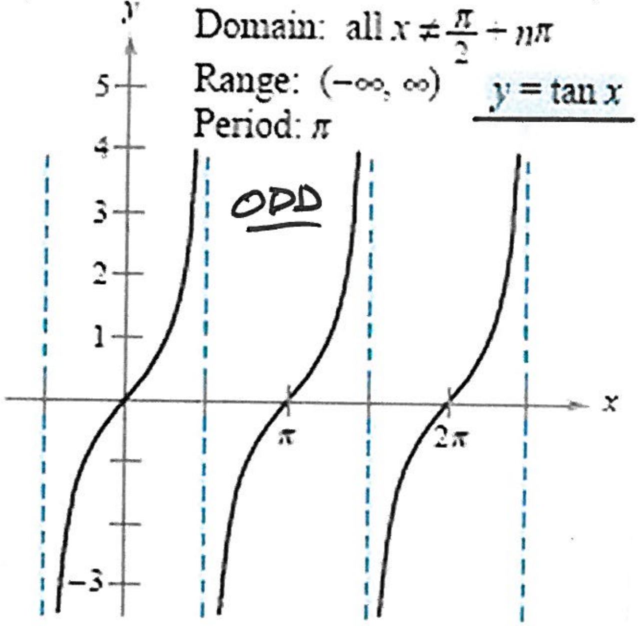
Domain: all reals
Range: [−1, 1]
Period: 2π

EVEN FUNCTION

$\cos(-x) = \cos x$

$y = \cos x$

$\tan(-x) = -\tan x$

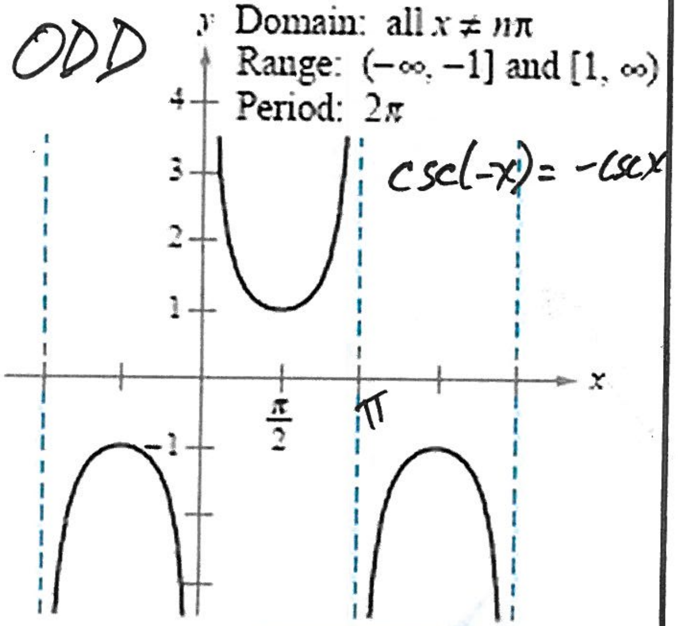


Domain: all $x \neq \frac{\pi}{2} + n\pi$
Range: $(-\infty, \infty)$
Period: π

ODD

$y = \tan x$

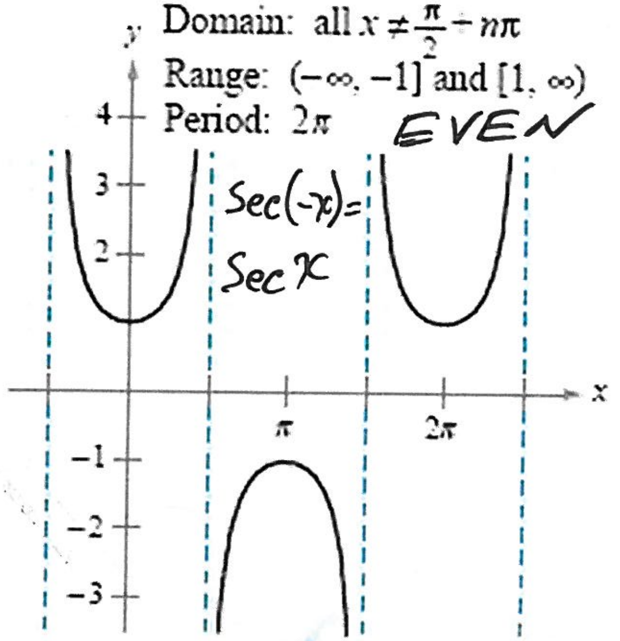
ODD



Domain: all $x \neq n\pi$
Range: $(-\infty, -1]$ and $[1, \infty)$
Period: 2π

$\csc(-x) = -\csc x$

$y = \csc x = \frac{1}{\sin x}$

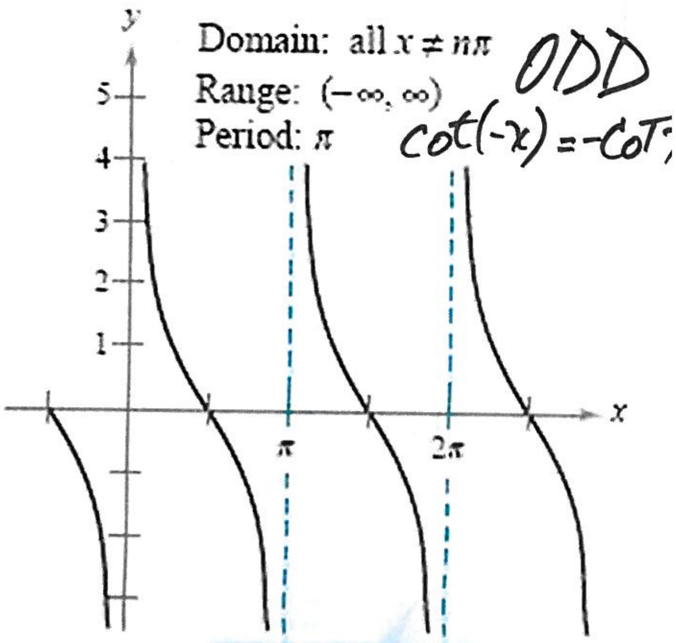


Domain: all $x \neq \frac{\pi}{2} + n\pi$
Range: $(-\infty, -1]$ and $[1, \infty)$
Period: 2π

EVEN

$\sec(-x) = \sec x$

$y = \sec x = \frac{1}{\cos x}$



Domain: all $x \neq n\pi$
Range: $(-\infty, \infty)$
Period: π

ODD

$\cot(-x) = -\cot$

$y = \cot x = \frac{1}{\tan x}$

# INVERSE Trig FUNCTIONS

IN order for a function to have an inverse, it must be one to one. ONE to ONE functions have one and only one unique corresponding value for every independent value ($x$)

To test whether a function is one to one, draw any horizontal line thru the graph of the function. If the line crosses more than one point on the graph, then it is NOT one to one.

As you can tell from the graphs of trig functions, they are not one to one. However, if you restrict their domains, you can make them one to one.

If $\sin\theta = x$, then

$$\theta = \arcsin x = \sin^{-1} x$$

Domain restricted $-1 \le x \le 1$

RANGE $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sin^{-1} x$ is read "The angle whose sine is $x$."

Example: $\sin^{-1}\left(-\frac{1}{2}\right) \Rightarrow$ The angle whose sine is $-\frac{1}{2}$ is $-\frac{\pi}{6}$

# Cofunction Identities

These are important when dealing with complementary angles. From the graphs you can see that the sine and cosine graphs look the same, but they have been shifted. This is where cofunction identities come in to play.

$$\sin\theta = \cos(90^\circ - \theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \sin(90^\circ - \theta) = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \cot(90^\circ - \theta) = \cot\left(\frac{\pi}{2} - \theta\right)$$

$$\sec\theta = \csc(90^\circ - \theta) = \csc\left(\frac{\pi}{2} - \theta\right)$$

$$\csc\theta = \sec(90^\circ - \theta) = \sec\left(\frac{\pi}{2} - \theta\right)$$

$$\cot\theta = \tan(90^\circ - \theta) = \tan\left(\frac{\pi}{2} - \theta\right)$$

# Periodicity Identities

The periods of sine, cosine, secant and cosecant repeat every $2\pi$ or 360°. Tangent and cotangent repeat every $\pi$ or 180°.

$$\sin(\theta + 2\pi) = \sin\theta \qquad \text{Cosecant}(\theta + 2\pi) = \csc\theta$$

$$\cos(\theta + 2\pi) = \cos\theta \qquad \text{Secant}(\theta + 2\pi) = \sec\theta$$

$$\tan(\theta + \pi) = \tan\theta \qquad \text{Cotangent}(\theta + 2\pi) = \cot\theta$$

# Trig Identities

$(\sin\theta)(\csc\theta) = 1$ because $\frac{y}{R} \bullet \frac{R}{y} = 1$

$(\cos\theta)(\sec\theta) = 1$ because $\frac{x}{R} \bullet \frac{R}{x} = 1$

$(\tan\theta)(\cot\theta) = 1$ because $\frac{y}{x} \bullet \frac{x}{y} = 1$

## Pythagorean Identities

Note: $\sqrt{\sin^2\theta} = \sin\theta$ And $\sin^2\theta = (\sin\theta)^2$

$\sin^2\theta + \cos^2\theta = 1$

So, $\cos^2\theta = 1 - \sin^2\theta$ and $\sin^2\theta = 1 - \cos^2\theta$

$\tan^2\theta + 1 = \sec^2\theta = \frac{1}{\cos^2\theta}$

$\cot^2\theta + 1 = \csc^2\theta$

## Sum and Difference Identities

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

# Additional Trig Identities

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

Note: $\sin\frac{\theta}{2} \neq \frac{\sin\theta}{2}$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta}$$
$$= \frac{1-\cos\theta}{\sin\theta}$$

ACT TEST Example:

If $\sin A = x$, which of the following is equal to $x$ for all points for which it is defined?

A) $1 - \cos A$ B) $(\cot A)(\cos A)$

c) $(\tan A)(\cos A)$ D) $\cos A - 1$

E) $\dfrac{\sec A}{\tan A}$

ANSWER C Since $\tan A = \dfrac{\sin A}{\cos A}$, then

$$\left(\frac{\sin A}{\cos A}\right)\left(\cos A\right) = \sin A = x$$

by substitution

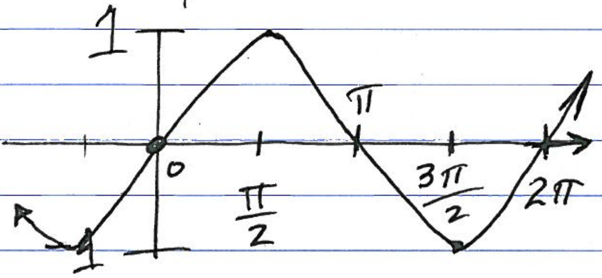
Answer c is the only answer that can equal Sin A

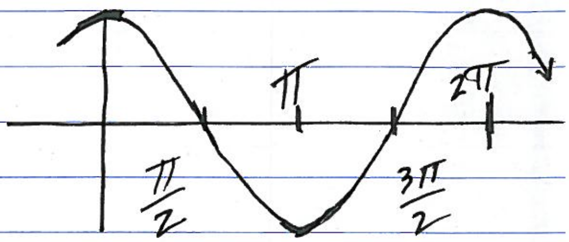
# Interpreting graphs of Trig Functions

General Trig Form $y = A\sin(Bx - C) + D$
or
$y = A\cos(Bx - C) + D$

You need to know how the constants A, B, C and D affect the basic graphs.

$y = \sin x$ $\qquad$ $y = \cos x$





Constants A and B change the SHAPE

Constants C and D change the POSITION

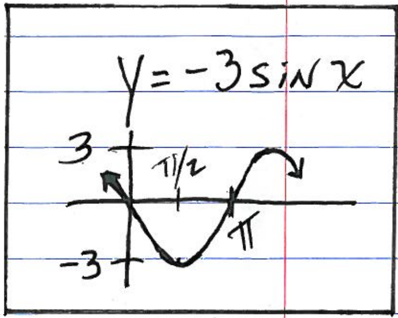
## CONSTANT A

- Changes the height or amplitude of graph
- Creates a vertical stretch or shrink
- determines how tall or short a graph is
- If $A > 1$, graph is taller or stretched
- if $A < 1$, graph is shrunk or compressed
- As A gets closer to 0, $y = \sin x$ gets closer to $y = 0$.
- If $A < 0$, then the graph is "upside down"

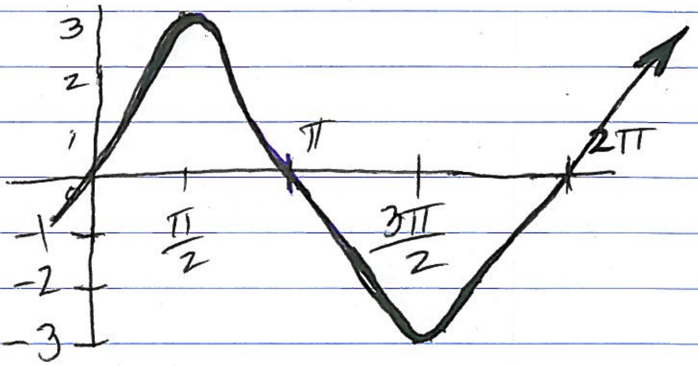
(Continued on Next page)

# CONSTANT A (cont.)

- $|A| = \text{Amplitude} = \dfrac{\text{max} - \text{min}}{2}$
- Amplitude is a <u>direct</u> relationship. An increase in A produces an increase in amplitude



y = 3 sin x



## Constant B in form $y = A\sin(Bx - c) + D$

- Constant B determines the PERIOD or the duration of x values between the max ~~or~~ min values of the waves.
- Period is defined as the time required for 1 complete regularly repeated process.
- Period is <u>horizontal</u> stretch or shrink
- If the graph were like a string, changes in B would be like pulling or pushing the L and R ends of the graph.
- B determines how wide or narrow a graph is
- <u>INDIRECT</u> relation. Increases in B create a decrease in period.
  - If $B > 1$, then function completes cycle quicker
  - If $B < 1$, then function completes cycle slower

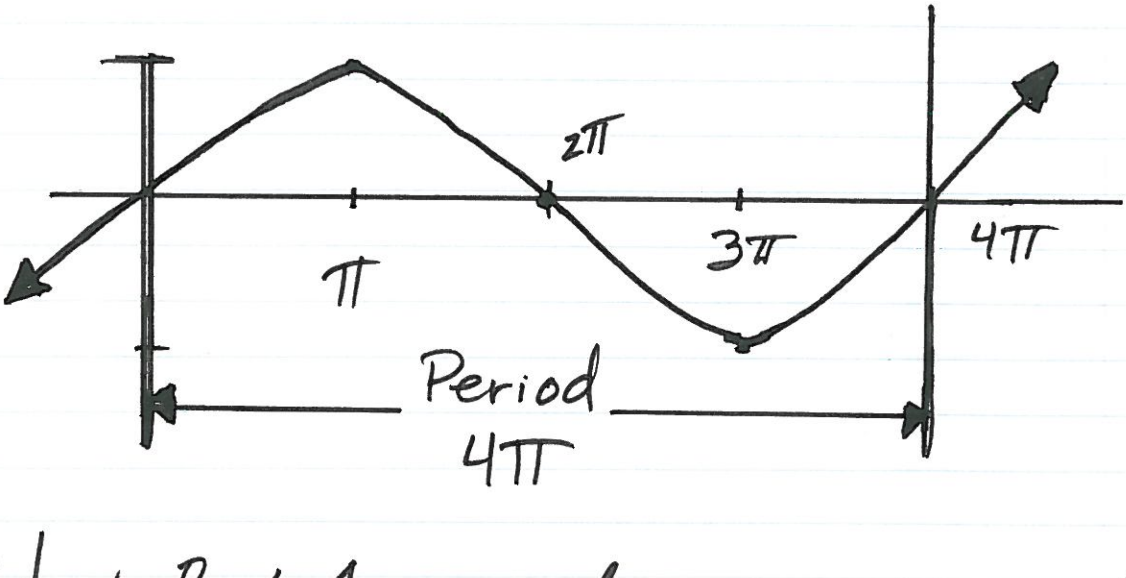
$$\text{Period} = \frac{360^\circ}{B} = \frac{2\pi}{B}$$

Standard period when B = $\underline{1}$. If period is greater than one, the graph is horizontally stretched.

The equation $Y = \sin \frac{1}{2}x$ illustrates how the period stretches the graph and how period is an INVERSE relationship

$$Y = \sin \tfrac{1}{2}x \qquad \text{Period} = \frac{2\pi}{B} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

$$B = \tfrac{1}{2}$$

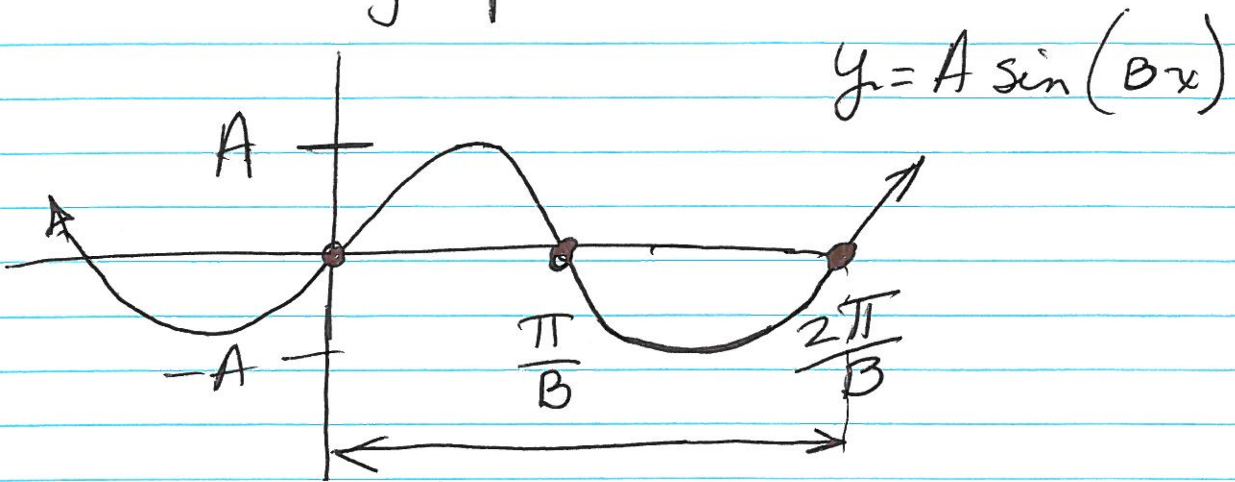


When B < $\underline{1}$ period INCREASES and cycle takes longER TO COmplETE

When B > 1 period decreases and cycle takes shorter to complete

# Constant B

Generic graph



$$\text{Period} = \frac{2\pi}{B}$$

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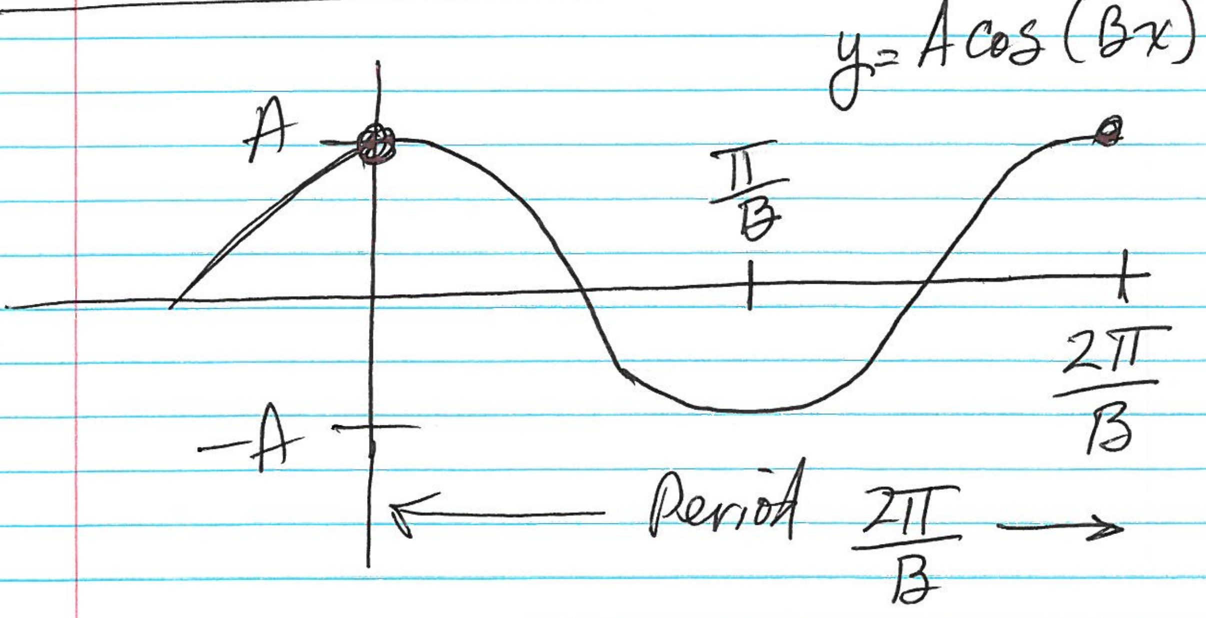


# The Constant C in the Equation $Y = A\sin(Bx - C) + D$

C represents a change in degrees or radians

C does not change the basic shape of the graph

C changes the horizontal position of the graph. It is a horizontal translation called a <u>PHASE SHIFT</u>

C is considered positive when in the form $Y = A\sin(Bx - c) + D$

Phase shift $= \dfrac{c}{B}$ = Starting point of graph

Graph starts at the point $x = \dfrac{c}{B}$

when $\dfrac{c}{B} > 0$, there is a phase shift to the right

When $\dfrac{c}{B} < 0$, phase shift is to the left

---

Example: $4\sin\left(2x - \dfrac{2\pi}{3}\right)$ $A = 4$ $B = 2$ $C = \dfrac{2\pi}{3}$

Amplitude $= |4| = 4$

period $= \dfrac{2\pi}{2} = \pi$

graph starts @ $x = \dfrac{\pi}{3}$

Phase shift $= \dfrac{\frac{2\pi}{3}}{2} = \dfrac{\pi}{3}$

# $y = A \sin(Bx - C) + D$ form

## CONSTANT D

- If C represented a left/Right change in position, then D represents an up/down change in position

- C makes a shift on the x-axis, D makes a shift on the y-axis.

- Notice the general form has "+" D. Therefore, if $D > 0$, then the translation is up from $Y = 0$.

- If $D < 0$, then the graph moves down from $Y = 0$ that many units.

## Summary of $Y = A \sin(Bx - C) + D$

Amplitude $= |A|$ if $A > 0$, then graph is right side up. If $A < 0$, then graph is upside down.

Period $= \dfrac{2\pi}{B} = \dfrac{360°}{B} =$ one complete cycle

Horizontal Phase Shift Left to right $= \dfrac{C}{B}$

Vertical Shift = up or down the Y-axis if $D > 0$ up, $D < 0$ down from $y = 0$

Example of graphing a function in the form of

$$y = A\cos(Bx - C)$$

Example Function $\quad y = 5\cos(4x + \pi)$

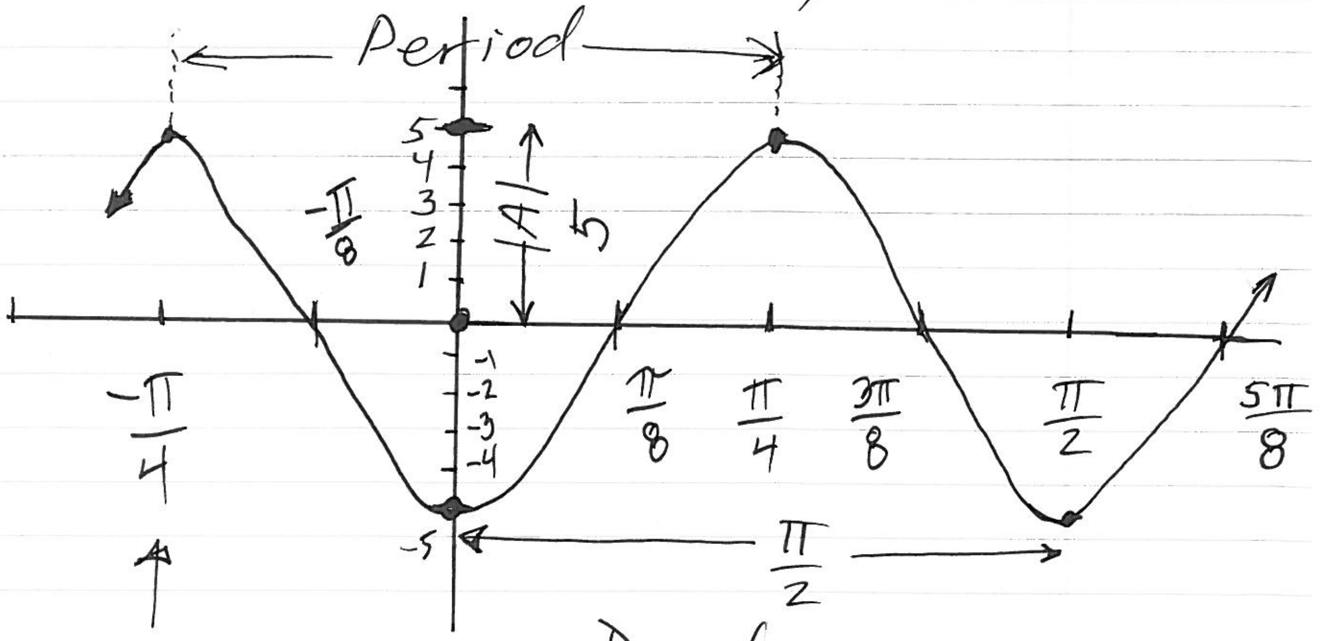
$A = 5 \quad B = 4 \quad C = -\pi$

$\text{Amplitude} = |A| = |5| = 5$

$\text{Period} = \dfrac{2\pi}{B} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

$\text{Phase Shift} = \dfrac{C}{B} = \dfrac{-\pi}{4}$

Graph Starts @ $x = \dfrac{-\pi}{4}$



graph
STARTS here

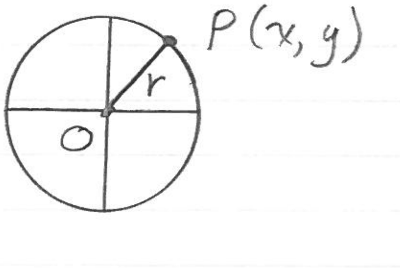
$\text{Period} = \dfrac{\pi}{2}$

# Conic Sections

The STANDARD Equation of a Circle is

$x^2 + y^2 = r^2$ for center at $(0,0)$
and $r$ = radius.

Proof uses distance formula $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$



Line $OP = r = \sqrt{(x-0)^2+(y-0)^2}$

$$r = \sqrt{x^2+y^2}$$

$$r^2 = x^2+y^2$$

When Center is Not (0,0) We use (h, k)
as the center and the formula is

$$(x-h)^2+(y-k)^2 = r^2$$

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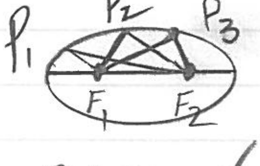
## Ellipses

An ellipse is the set of all points P in a plane such that the sum of the distances from P to Two fixed points $F_1$ and $F_2$ called the foci, is a constant.



$\overline{F_1P} + \overline{F_2P}$ = CONSTANT

AS P MOVES around the ellipse

$$\overline{P_1F_1} + \overline{P_1F_2} = \overline{P_2F_1} + \overline{P_2F_2} = \overline{P_3F_1} + \overline{P_3F_2}$$

The sum of the distances from any point on the outside of the ellipse To each of the foci is CONSTANT

# Proof of ellipse formula

You can formulate the equation for any ellipse using the distance formula $\rightarrow$ $\overline{xy} = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

From the Above ellipse, $F_1=(-4,0)$ $F_2=(4,0)$ and say $\overline{F_1P_1} + \overline{F_2P_1} = 10$. Substituting, we ge

$$\sqrt{(x-(-4))^2+(y-0)^2} + \sqrt{(x-4)^2+(y-0)^2} = 10$$

$$\sqrt{(x+4)^2+y^2} = 10 - \sqrt{(x-4)^2+y^2}$$

Square both sides $(x+4)^2+y^2 = 100 - 20\sqrt{(x-4)^2+y^2} + [(x-4)^2+y^2]$

$$x^2+8x+16+y^2 = 100 - 20\sqrt{(x-4)^2+y^2} + [x^2-8x+16+y^2$$

Simplify $16x - 100 = -20\sqrt{(x-4)^2+y^2}$

Divide out 4 $4x - 25 = -5\sqrt{(x-4)^2+y^2}$

Square Again $16x^2-200x+625 = 25[(x-4)^2+y^2]$

$$16x^2-200x+625 = 25x^2-200x+400+25y^2$$

$$225 = 9x^2+25y^2$$

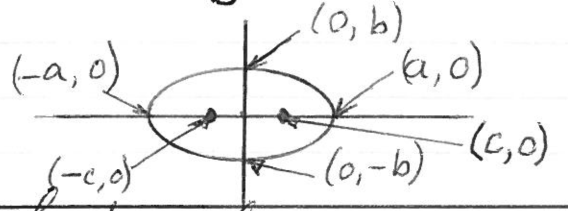
Divide each side by 225 $1 = \frac{x^2}{25} + \frac{y^2}{9}$

This exercise illustrates how the general equation is formed.

# Standard equation for ellipse When center is the origin (0, 0)

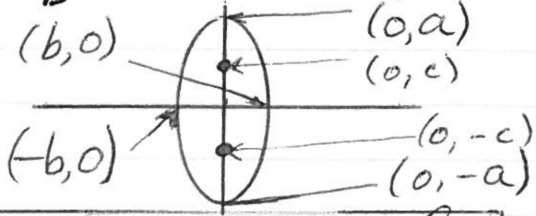
Horizontal Major axis

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Vertical MAJOR AXIS

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



The length of major axis = 2a and the length of minor axis is 2b
IN each case, $a^2 > b^2$ and $a^2 - b^2 = c^2$

When the origin is Not the center, The formula is the following for horizontal

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

If center is (2, 3), then $h = 2$ and $k = 3$

For Vertical Axis ellipse, the formula is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

"h" represents the Amount of translation from the origin pertaining to "x"

"k" represents the amount of translation from the origin pertaining to "y"

# Completing the Square

Indirectly Needed to analyze quadratics, parabolas, circles, graphs for ellipses and hyperbola's: GENERAL FORM is

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Completing the square is taking any trinomial and creating a binomial squared. HERE'S a problem with Explanation.

$$2x^2 + 12x - 16 = 0$$

GENERAL FORM $Ax^2 + Bx - C = 0$

START All Equations with Trinomial Equal TO Zero.

1) Divide out A term so that $A = \underline{1}$

$$\frac{2x^2 + 12x - 16 = 0}{2} = x^2 + 6x - 8 = 0$$

2) Move C term to other side

$x^2 + 6x - 8 = 0$ becomes $x^2 + 6x = 8$

3) TAKE 1/2 of the B term, square it and add To both Sides.

$$\left(\frac{1}{2}(6)\right)^2 = 9 \Rightarrow x^2 + 6x + 9 = 8 + 9$$

4) Now you can create a perfect square.

$x^2 + 6x + 9 = 17$ becomes $(x + \underset{\uparrow}{3})^2 = 17$

Perfect Square